Stable Power Domains

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Abstract

In the category of stable dcpo's, free constructions w.r.t. algebraic theories exist. From this, we obtain various stable power domain constructions. After handling their properties in general, we concentrate on the *stable Plotkin power construction*. For continuous ground domains, it is explicitly described in terms of *saturated compact sets*. In case of algebraic ground domains, this description is isomorphic to Buneman's *lossless power domains*.

1 Introduction

In [3], Berry introduced the notion of *stability* to ban certain parallel functions like the 'parallel or' from the semantic domains of sequential languages. Although stability does not exclude all parallel functions, it can be understood as an approximation to the — not yet semantically describable — notion of sequentiality.

Berry introduced a special kind of domains — the *dI domains* — as the objects of his category of stable mappings. Like the classical category of Scott domains and continuous functions, the category of dI domains suffers from the fact that it is not small complete: equalizers of parallel pairs of morphisms do not exist in general. It is well known that the category of Scott domains can be embedded into the much larger category of all dcpo's that is both small complete and cartesian closed. In [1], Amadio presented a category of *stable dcpo's* and 'stable' mappings that is small complete and cartesian closed, and contains the dI domains as a full subcategory. At the end of his paper, Amadio asks whether stable power constructions exist. We now can answer this question: yes, they exist, but they differ much from their classical analogues.

In the previous paragraph, we put the word stable into quotes, because the stable mappings of Amadio are called *conditionally multiplicative* (*cm*) by Berry. Being *cm* is an approximation of the mathematically complex notion of stability in [3]. Whereas on dI domains,

stability and *cm* are equivalent notions, this is not true for more general kinds of domains. Nevertheless, we shall adopt Amadio's habit to call the *cm* functions stable (cf. Def. 2.1).

In Section 2, we introduce the category SCPO of stable dcpo's (scpo's) and stable maps. In Section 3, we indicate that it admits free constructions for algebraic theories. We also investigate such free constructions in general. In Section 4, this knowledge is applied to several power theories and the corresponding free constructions. It is shown that lower power constructions are degenerated, and upper constructions suffer from the fact that their extension functional is not monotonic. Thus, only the stable analogue $\mathcal{P}$ of the classical convex or Plotkin power construction is considered further.

So far, $\mathcal{P}X$ was implicitly characterized as the free stable semilattice over the scpo X. In the second part of the paper, our goal is to develop explicit descriptions of $\mathcal{P}X$ for certain classes of ground domains X. Without any hints how to proceed, it would be a difficult task to find an explicit description. Fortunately, the theory of the classical upper or Smyth power construction(s) in the category DCPO provides enough intuition how to obtain explicit stable power construction(s) in SCPO.

Upper power constructions in DCPO may be defined in the following ways [20, 7, 12, 11]:

(1) $\mathcal{U}_{\mathrm{i}}X$ is defined as the free $\mathcal{U}$-algebra in DCPO over the dcpo X, as proposed in [13]. This is analogous to what we do in the first part of this paper.

(2) A functional definition: Let $\mathcal{U}_{\mathrm{f}}X = [[X \to \mathbf{U}] \stackrel{add}{\to} \mathbf{U}]$ be the set of all additive non-empty 'second-order predicates' over X. Here, $\mathbf{U}$ is $\{1, 0\}$ with $1 \sqsubset 0$. Additive means $A(\lambda x.\, 0) = 0$ and $A(\lambda x.\, px + qx) = Ap + Aq$, and non-empty means $A(\lambda x.\, 1) = 1$. As we are in DCPO, the function spaces are equipped with the pointwise order.

(3) Let $\mathcal{U}_{\Phi}X$ be the set of all Scott open filters of Scott open sets of X ordered by inclusion, as defined in [20]. Since continuous functions to $\mathbf{U}$ correspond to Scott open sets, $\mathcal{U}_{\Phi}X$ and $\mathcal{U}_{\mathrm{f}}X$ are isomorphic for all dcpo's X.

(4) A topological definition: Let $\mathcal{U}_{\mathrm{k}}X$ be the set of all non-empty Scott compact upper sets of X ordered by inverse inclusion '$\supseteq$'. In [20], $\mathcal{U}_{\mathrm{k}}X$ and $\mathcal{U}_{\Phi}X$ are shown to be isomorphic iff X is sober in its Scott topology. In this case, the second-order predicate A corresponding to a non-empty compact upper set S is defined by $Ap = 1$ iff there is x in S with $px = 1$.

(5) The algebraic case: If X is algebraic, let $\mathcal{U}_{\mathrm{a}}X$ be the ideal completion of the poset of all non-empty finitely upper subsets of the basis of X, ordered by inverse inclusion. Here, finitely upper means $\uparrow E$ for some finite set E.

We already mentioned that $\mathcal{U}_{\mathrm{f}}X$, $\mathcal{U}_{\Phi}X$, and $\mathcal{U}_{\mathrm{k}}X$ are isomorphic for sober X. $\mathcal{U}_{\mathrm{k}}X$ and $\mathcal{U}_{\mathrm{i}}X$ are isomorphic for continuous X, whereas they differ for some non-continuous (but still sober) X [11]. In case of algebraic X — the only case where it is defined — $\mathcal{U}_{\mathrm{a}}X$ is isomorphic to the other upper power domains.

These results about classical upper constructions have led our search for explicit descriptions of $\mathcal{P}$. In Section 5, we introduce the basic results of 'stable topology' needed in the further development. In Section 6, we define a construction $\mathcal{P}_{\mathrm{f}}X$ in terms of second order predicates, and transform it into a filter representation $\mathcal{P}_{\Phi}X$ and a topological representation $\mathcal{P}_{\mathrm{k}}X$, provided that X is 'stably sober', which is satisfied by every continuous scpo. Under the assumption of continuity of X, we show in Section 7 that $\mathcal{P}_{\mathrm{k}}X$ is the free stable semilattice over X, i.e., $\mathcal{P}_{\mathrm{k}}X \cong \mathcal{P}X$. In Section 8, we turn to the algebraic case and derive from $\mathcal{P}_{\mathrm{k}}X$

a representation of $\mathcal{P}X$ via basis and ideal completion, which, surprisingly, coincides with the *lossless power domains* of [4, 14], which were proposed without any regard of stability or universal properties.

In Section 9, we consider various classes of scpo's, and investigate whether they are preserved by the stable power construction.

The present paper is a shortened and generalized version of the technical report [9]. It is more general, because the report shows $\mathcal{P}_{\mathrm{k}}X \cong \mathcal{P}X$ for a special class of continuous scpo's only. It is shortened, because we omitted the more obvious proofs as well as some more sophisticated proofs that are not on the main course of development. For the latter, we explicitly refer to the corresponding fact in the report.

2 The Categories DCPO and SCPO

In Subsection 2.1, we present some standard notations and the category DCPO of dcpo's and continuous functions. In 2.2, we introduce the small complete and cartesian closed category SCPO of stable dcpo's (*scpo's*) and stable maps. The notion of compatibility, which is essential for the definition of stability, is in Subsection 2.3 generalized to *weak compatibility*. Subsection 2.4 deals with the full subcategory of *separable* scpo's, which is still small complete and cartesian closed. When we apply topological methods in this paper, we usually have to restrict ourselves to separable scpo's.

2.1 Standard Notations

A *poset* (partially ordered set) is a set P together with a reflexive, antisymmetric, and transitive relation '$\sqsubseteq$'. We often identify the poset $(P, \sqsubseteq)$ with its carrier P.

For $A \subseteq P$, let $\downarrow A$ be the set of all points *below* some point of A, and correspondingly $\uparrow A$ the set of all points *above* some point of A. A set $A \subseteq P$ is a *lower set* iff $\downarrow A = A$, and an *upper set* iff $\uparrow A = A$.

We refer to the standard notions of upper bound, least upper bound or join (denoted by $\sqcup$), greatest lower bound or meet (denoted by $\sqcap$), directed set, and monotonic function. A function $f : P \to Q$ is an *order embedding* of P into Q iff $a \sqsubseteq b$ in P is equivalent to $fa \sqsubseteq fb$ in Q.

A *dcpo* is a poset where every directed set has a join. A dcpo need not have a least element. A particularly important dcpo is $\mathbf{2} = \{\bot, \top\}$ where $\bot \sqsubset \top$. A monotonic function $f : X \to Y$ between two dcpo's is *continuous* iff it preserves the joins of directed sets. It is well known that the category DCPO of dcpo's and continuous maps is small complete (i.e., has all limits) and cartesian closed (i.e., has finite products and exponentials). We refer to this category as 'classical'[1] in contrast to the 'stable' category to be introduced later.

If an upper set O has the property that $\bigsqcup D \in O$ implies $D \cap O \neq \emptyset$ for all directed sets D, then it is called *Scott open*. The Scott open sets of a dcpo X form a topology on X, the *Scott topology*.

[1] In view of the morphisms, 'continuous' would be more appropriate, but unfortunately, this notion is overloaded: it also describes a possible property of dcpo's.

In a dcpo, a point x is *way-below* a point y, in formulae $x \ll y$, if for all directed sets D, $y \sqsubseteq \bigsqcup D$ implies $x \sqsubseteq d$ for some d in D. A subset B of a dcpo X is a *basis* if for all x in X, the set $\{b \in B \mid b \ll x\}$ is directed with join x. A dcpo X is *continuous* iff it has at least one basis. For every continuous dcpo X, the whole carrier X is a basis.

In a dcpo, a point a is *isolated* iff it is way-below itself. A dcpo X is *algebraic* iff it has a basis of isolated points. This basis then consists of all isolated points, and is contained in every other basis. An algebraic dcpo can be recovered from its poset of isolated points by *ideal completion*.

2.2 Stability

Two points a and b of a poset are *compatible*, $a \uparrow b$, if they have a common upper bound. Stability is the requirement to respect meets of pairs of compatible points.

Definition 2.1 A dcpo X is *stable* iff every pair of compatible points $a \uparrow b$ of X has a meet $a \sqcap b$, and the meet operation is continuous for compatible points: if $D \subseteq X$ is directed and $x \uparrow \bigsqcup D$, then $x \sqcap \bigsqcup D = \bigsqcup_{d \in D} x \sqcap d$. We abbreviate the wording stable dcpo to *scpo*.

A function $f : X \to Y$ between two scpo's is *stable* iff it is continuous and respects compatible meets: $a \uparrow b$ implies $f(a \sqcap b) = fa \sqcap fb$. The category of scpo's and stable functions is called SCPO.

Note that our definition of scpo's and stable functions equals that in [1] except for the slight change that we do not require a bottom element in the scpo's. This is important for the existence of equalizers.

Every set induces an scpo when ordered by $x \sqsubseteq y$ iff $x = y$. Such scpo's are called *discrete*. Every map from a discrete scpo to an arbitrary scpo is stable.

Theorem 2.2 The category SCPO is small complete and cartesian closed. The product of a family $(X_i)_{i \in I}$ of scpo's is the product of the carrier sets of the X_i with order $(x_i)_{i \in I} \sqsubseteq (x'_i)_{i \in I}$ iff $x_i \sqsubseteq x'_i$ for all i in I. The equalizer of a parallel pair of stable functions $f, g : X \to Y$ is $\{x \in X \mid fx = gx\}$ with order inherited from X. The exponential $[X \to Y]$ consists of the stable functions from X to Y ordered by the *stable order*:

$$f \sqsubseteq g \text{ iff } \forall x, x' \in X : (x \sqsubseteq x' \text{ implies } fx = fx' \sqcap gx)$$

The stable order implies the pointwise order, i.e., $f \sqsubseteq g$ implies $fx \sqsubseteq gx$ for all x in X. Directed joins and compatible meets are given pointwise in $[X \to Y]$: $(\bigsqcup_{i \in I} f_i)\, x = \bigsqcup_{i \in I} f_i x$ and $(f \sqcap g)\, x = fx \sqcap gx$.

Proof: Cartesian closedness is shown in [1]. The treatment of finite products can be easily extended to general products. The verification of the claimed equalizers is straightforward.□

The main difference to the classical case is that functions are not ordered pointwise. Thus, the exponential $[X \to Y]$ cannot be canonically embedded into the product $\prod_{x \in X} Y$.

In [17, 2], it is shown that the simply typed λ-calculus can be interpreted in any cartesian closed category so that types denote objects and λ-expressions denote arrows. In the sequel, we identify λ-expressions and arrows, types and objects. For instance, $\lambda x^X.\, x$ is the identity

id_X on object X, and $\lambda x^X.\, g(fx)$ is the composition of $f : X \to Y$ and $g : Y \to Z$. The type superscripts at variables will often be dropped. The fact that simply typed λ-calculus can be interpreted in the category SCPO then means that every well-typed closed λ-expression that is built from stable functions is a stable function again. We shall use this fact to avoid many explicit proofs of the stability of particular functions.

2.3 Weak Compatibility

Both scpo's and stable maps are defined using the compatibility relation '$\uparrow$'. However, this relation has the drawback that it is not 'continuous' in general, i.e., it is not closed under directed joins. Consider the following example found in [14]:

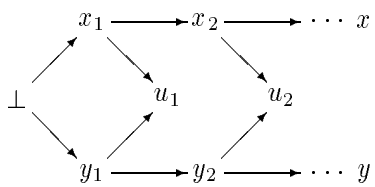


Figure 1

This is an algebraic scpo, where $x_n \uparrow y_n$ via u_n for all n, but the limit points x and y are not compatible.

In this subsection, we show how this problem can be overcome by introducing *weak compatibility*. With $\Delta = \Delta_X = \{(x, x) \mid x \in X\}$, it is easily seen that $u \uparrow v$ iff $(u, v) \in \downarrow\Delta$. This leads to the definition that u and v are weakly compatible, $u \,\overline{\uparrow}\, v$, iff $(u, v) \in \mathsf{cl}\,\Delta$, where '$\mathsf{cl}$' denotes Scott closure.

By the definition of scpo's, only meets of compatible points are required to exist. This can be generalized to weakly compatible points.

Proposition 2.3 In every scpo, meets of weakly compatible points exist and the meet operation is continuous. In formulae: $\sqcap : \mathsf{cl}\,\Delta_X \to X$ is defined and continuous for every scpo X.

Proof: By Prop. 2.4 in the report [9], the set $D_\sqcap$ of all pairs, whose meet exists, is Scott closed in X^2, and $\sqcap : D_\sqcap \to X$ is continuous. Since $\downarrow\Delta_X \subseteq D_\sqcap$ and $D_\sqcap$ is closed, $\mathsf{cl}\,\Delta_X \subseteq D_\sqcap$ follows. □

We now present some properties of weak compatibility.

Proposition 2.4

(1) If $x \,\overline{\uparrow}\, x'$ in X and $y \,\overline{\uparrow}\, y'$ in Y, then $(x, y) \,\overline{\uparrow}\, (x', y')$ in $X \times Y$.

(2) Let $f : X \to Y$ be continuous. If $x \,\overline{\uparrow}\, x'$ in X, then $fx \,\overline{\uparrow}\, fx'$ in Y.

(3) Let $f : X \to Y$ be stable, and let $x \,\overline{\uparrow}\, x'$ in X. Then $f(x \sqcap x') = fx \sqcap fx'$ holds.

Proof: Parts (1) and (2) are shown by straightforward (Scott) topological arguments. For (3), the set $\{(x, x') \in \mathsf{cl}\,\Delta_X \mid f(x \sqcap x') = fx \sqcap fx'\}$ is shown to be closed. □

By Prop. 2.3 and part (3) of Prop. 2.4, compatibility could be replaced by weak compatibility in Definition 2.1 without changing the notions of scpo and stable function.

2.4 Separability

Now, we define *separability* as a possible property of scpo's. In the remainder of this paper, we shall often meet statements that can be proved for separable scpo's, but not for general ones.

Definition 2.5 An scpo X is called *separable* iff for all points a and b of X with $a \not\sqsubseteq b$, there is a stable map $\sigma : X \to \mathbf{2}$ with $\sigma a \not\sqsubseteq \sigma b$, i.e., $\sigma a = \top$ and $\sigma b = \bot$. We say that σ separates a from b.

In the classical case, all dcpo's are separable: if $a \not\sqsubseteq b$, then mapping all points below b to $\bot$ and all other points to $\top$ is a continuous map $h : X \to \mathbf{2}$ with $ha = \top$ and $hb = \bot$. Stability of this map is not guaranteed however, whence the situation in case of SCPO is more complex. There are separable scpo's as well as non-separable ones.

In Subsection 7.1, we shall prove that *every continuous scpo is separable* (Theorem 7.2). Here, continuity of an scpo means continuity in the usual sense of the underlying dcpo. Thus, all the scpo's that occur in common semantic theories will be separable.

On the other hand, an example of a non-separable scpo is given by the set **RC** of *regular closed* subsets of the interval [0..1] of the real numbers, ordered by set inclusion. **RC** was introduced in [5] as an example of an atomless complete Boolean algebra. It has uncountably many elements, but the only stable maps from **RC** to **2** are the two constant maps $\lambda x.\, \bot$ and $\lambda x.\, \top$, which cannot separate anything.

Problem 1 Is there a universal separator S such that replacing **2** by S in the definition above would make all scpo's separable?

Separability has the following properties:

Theorem 2.6

(1) Products of separable scpo's are separable.

(2) Sub-scpo's of separable scpo's are separable.

(3) If Y is separable, then $[X \to Y]$ is separable.

Proof: The proofs of (1) and (2) are straightforward, whereas (3) is a bit involved because of the stable order. This is Theorem 2.11 in the report [9]. □

By this theorem, the category of separable scpo's is also small complete and cartesian closed.

3 Free Constructions on Stable Dcpo's

Since we want to define stable power constructions as free constructions w.r.t. certain algebraic theories, we first investigate these constructions in general. Theories and free constructions are introduced in 3.1. In 3.2, we show that under a mild hypothesis, the generator function is an order embedding for separable ground domains. In 3.3, we present a criterion for the stability of the extension functional of free constructions. Stability holds if the corresponding theory is *exponentiable*, i.e., can be raised to function spaces by abstraction.

3.1 Algebraic Theories and Free Constructions

Usually, an algebraic theory consists of a set of operators with given arity and a set of axioms in form of equations $L = R$ over these operators. Since we want to consider algebraic theories in the category SCPO, we also allow inequations $L \sqsubseteq R$ as axioms. Algebraic theories with inequations are needed for the lower and upper power domain constructions.

A *model* of an algebraic theory $\mathcal{T}$ in the category SCPO, or shortly a *$\mathcal{T}$-algebra*, is an scpo — the *carrier* — together with a set of *stable* functions — the *operations* — interpreting the operators and satisfying the axioms. The functions interpreting operators of arity n have type $X^n \to X$ where X is the carrier. A $\mathcal{T}$-algebra homomorphism, or shortly *$\mathcal{T}$-morphism*, is a *stable* function between the carriers of $\mathcal{T}$-algebras preserving all operations.

There is an obvious forgetful functor from the category of $\mathcal{T}$-algebras and $\mathcal{T}$-morphisms to SCPO. We do not make this functor explicit, but identify a $\mathcal{T}$-algebra with its carrier, thus speaking of e.g. stable functions $f : X \to A$ from an scpo X to a $\mathcal{T}$-algebra A.

Definition 3.1 (Free constructions) A *free construction* $\mathcal{T} = (\mathcal{T}, \mathsf{s}, \mathsf{E})$ for a theory $\mathcal{T}$ maps every scpo X into a $\mathcal{T}$-algebra $\mathcal{T}X$ such that there is a stable function $\mathsf{s} : X \to \mathcal{T}X$, and for every $\mathcal{T}$-algebra A and every stable function $f : X \to A$, there is a unique $\mathcal{T}$-morphism $\mathsf{E}f : \mathcal{T}X \to A$ extending f, i.e., $\mathsf{E}f \circ \mathsf{s} = f$. $\mathcal{T}X$ is called the *free $\mathcal{T}$-algebra* over X. The elements $\mathsf{s}x$ with x in X are its *generators*.

As can be seen from the definition, we adopt the convention to denote a free construction by the same symbol as its algebraic theory. Categorically speaking, free constructions for $\mathcal{T}$ are left adjoint to the forgetful functor from $\mathcal{T}$-algebras to scpo's. Free constructions are uniquely determined up to isomorphism. The most important fact is the existence of free constructions:

Theorem 3.2 In the category SCPO, free constructions exist for all algebraic theories.

Proof: This is Theorem 2.7 in the report [9]. □

The following two subsections will be concerned with two general questions:

(1) Is the generator map $\mathsf{s} : X \to \mathcal{T}X$ an order embedding?

(2) Is the extension functional $\mathsf{E} : [X \to A] \to [\mathcal{T}X \to A]$ stable?

We shall provide sufficient conditions to answer both questions with 'yes'. In doing so, we will find some obstacles that do not exist in the classical case.

3.2 Non-Degeneration

First, we tackle question (1) above. We show that s is an order embedding if there are non-discrete $\mathcal{T}$-algebras and X is separable. An scpo is *discrete* iff $a \sqsubseteq b$ implies $a = b$. Consequently, it is non-discrete iff there are points a and b such that $a \sqsubset b$, i.e., $a \sqsubseteq b$ and $a \neq b$.

Theorem 3.3 If there is at least one non-discrete $\mathcal{T}$-algebra, then $\mathsf{s} : X \to \mathcal{T}X$ is an order embedding for all separable scpo's X. Otherwise for all non-discrete X, s is not injective.

Proof: Let A be a non-discrete $\mathcal{T}$-algebra, and let u, v in A with $u \sqsubset v$. Let X be a separable scpo and let $\mathsf{s}x \sqsubseteq \mathsf{s}x'$ for two points x and x' of X. We have to show $x \sqsubseteq x'$.

Assuming the contrary, there is a stable map $\sigma : X \to \mathbf{2}$ such that $\sigma x = \top$ and $\sigma x' = \bot$. Mapping $\bot$ to u and $\top$ to v forms a stable map $\tau : \mathbf{2} \to A$. The composition $f = \tau \circ \sigma : X \to A$ is stable, and maps x to v and x' to u. Its extension $\mathsf{E}f : \mathcal{T}X \to A$ maps $\mathsf{E}f\,(\mathsf{s}x) = fx = v$ and $\mathsf{E}f\,(\mathsf{s}x') = u$ contradicting $\mathsf{s}x \sqsubseteq \mathsf{s}x'$.

Finally, consider the 'otherwise' case: all $\mathcal{T}$-algebras are discrete. If X is non-discrete, then there are points $x \sqsubset x'$ in X. Then $\mathsf{s}x \sqsubseteq \mathsf{s}x'$ follows, whence $\mathsf{s}x = \mathsf{s}x'$. Thus, s is not injective. □

3.3 Stability of Extension

In this subsection, we present conditions for the stability of the extension functional. Following [18], stability of extension ('functorial strength') is equivalent to the 'tensorial strength' of the monad induced by the free construction. This 'strength' is necessary in Moggi's semantic framework. Practically speaking, a stable extension allows the derivation of a host of further stable functions.

An important example of a function derived from E and s is the mapping functional M. The extension functional has particular instances $\mathsf{E} : [X \to \mathcal{T}Y] \to [\mathcal{T}X \to \mathcal{T}Y]$. By defining $\mathsf{M}f = \mathsf{E}\,(\mathsf{s} \circ f)$, one obtains mapping functionals $\mathsf{M} : [X \to Y] \to [\mathcal{T}X \to \mathcal{T}Y]$. These functionals can be shown to be functorial: $\mathsf{M}\ id = id$ and $\mathsf{M}\,(g \circ f) = \mathsf{M}g \circ \mathsf{M}f$. The generator functions then become a natural transformation because of $\mathsf{M}f \circ \mathsf{s} = \mathsf{E}\,(\mathsf{s} \circ f) \circ \mathsf{s} = \mathsf{s} \circ f$. Stability of E directly implies stability of M, which is needed to show for instance the closedness of the class of stably bifinite scpo's under certain power constructions (see Theorem 9.7).

In the sequel, we shall show by means of the λ-calculus that extension is stable whenever the underlying algebraic theory can cope properly with function spaces.

Definition 3.4 An algebraic theory $\mathcal{T}$ is *exponentiable* iff for every scpo X and $\mathcal{T}$-algebra A, the function space $[X \to A]$ becomes a $\mathcal{T}$-algebra, if the operations op_A on A are raised to operations $op_{[X \to A]}$ on functions defined pointwise: $op_{[X \to A]}\,(f_1, \ldots, f_n) = \lambda x^X.\, op_A\,(f_1 x, \ldots, f_n x)$.

In the classical case, all theories are exponentiable: since both equality and order of functions are defined pointwise, the validity of the axioms on A implies the validity on $[X \to A]$. In the stable case, the situation becomes more complex because functions are not ordered pointwise. Thus, certain inequational axioms cannot be lifted to the function space, whereas others can be. Examples are provided by the axioms of the various power theories introduced in Section 4.

Theorem 3.5 Free constructions for exponentiable theories have stable extension.

Proof: Since it is too complex to prove stability of E directly, we construct a stable function $\widetilde{\mathsf{E}}$ and show that it equals E.

Let us consider the closed λ-expression $a \equiv \lambda x^X.\lambda f^{[X \to A]}.fx$ with type $X \to [[X \to A] \to A]$. By exponentiability, $[[X \to A] \to A]$ is a $\mathcal{T}$-algebra. Thus, a has a stable extension $\mathsf{E}a : \mathcal{T}X \to [[X \to A] \to A]$. By rearranging the arguments of $\mathsf{E}a$, we obtain

$\widetilde{\mathsf{E}} \equiv \lambda f^{[X \to A]}.\lambda u^{\mathcal{T}X}.\,\mathsf{E}\,a\ u\ f : [X \to A] \to [\mathcal{T}X \to A]$.

The function $\widetilde{\mathsf{E}}$ is stable because it is built from stable functions by means of the λ-calculus. To complete the proof, we have to show $\widetilde{\mathsf{E}}f = \mathsf{E}f$ for all stable $f : X \to A$. This is done by the uniqueness assertion of freeness. The right hand side is a $\mathcal{T}$-morphism by definition of $\mathsf{E}f$. The left hand side is a $\mathcal{T}$-morphism since for every operation $op_{\mathcal{T}X}$,

$$\begin{aligned}\widetilde{\mathsf{E}}f\,(op_{\mathcal{T}X}\,(x_1,\,\ldots,\,x_n)) &= \mathsf{E}a\,(op_{\mathcal{T}X}\,(x_1,\,\ldots,\,x_n))\,f\\ &= op_{[[X\to A]\to A]}\,(\mathsf{E}a\,x_1,\,\ldots,\,\mathsf{E}a\,x_n)f\\ &= op_A\,(\mathsf{E}a\,x_1 f,\,\ldots,\,\mathsf{E}a\,x_n f)\\ &= op_A\,(\widetilde{\mathsf{E}}f\,x_1,\,\ldots,\,\widetilde{\mathsf{E}}f\,x_n).\end{aligned}$$

Here, we used the fact that $\mathsf{E}a$ is a $\mathcal{T}$-morphism as the extension of a. Next, we show that $\widetilde{\mathsf{E}}f$ and $\mathsf{E}f$ coincide on generators. $\mathsf{E}f\,(\mathsf{s}x)$ yields fx, and

$$\widetilde{\mathsf{E}}f\,(\mathsf{s}x) \;=\; \mathsf{E}a\,(\mathsf{s}x)\,f \;=\; a\,x\,f \;=\; fx$$

shows that $\widetilde{\mathsf{E}}f\,(\mathsf{s}x)$ yields fx, too. □

4 Power Theories and Free Power Constructions

In this section, we introduce power domain constructions as free constructions w.r.t. certain algebraic theories, and investigate their properties. All our power theories contain a binary operation that is commutative, associative, and idempotent. We show that stability imposes severe restrictions on such operations. Then, we specialize the results of the previous section: we investigate which power constructions properly embed the ground domain into the power domain, and consider stability of the extension functional. After this, we show how the power domains of different theories are related to each other. Finally, we treat a particularly simple case of scpo's, where the power domains can be constructed explicitly.

The plain power theory $\mathcal{P}$ has just one binary operation '+' that is commutative, associative, and idempotent:

$$+ : A \times A \to A \qquad a + b = b + a \qquad a + (b + c) = (a + b) + c \qquad a + a = a$$

$\mathcal{P}$-algebras are also called *semilattices*. In every semilattice, there is a derived order defined by $a \leq b$ iff $a + b = b$. In general, this order differs from the order '$\sqsubseteq$' of the underlying scpo. The derived order '$\leq$' will not be used in this paper.

The free construction for theory $\mathcal{P}$ is called *convex* or *Plotkin* power construction in the classical case. The name *Vietoris* construction is also in use. In the stable case, there is nothing convex in this construction.

The theory $\mathcal{P}^0$ has a neutral element for '+': $\quad 0 : A \quad a + 0 = a$

The theory $\mathcal{P}_\perp$ has a distinguished least element: $\quad \perp : A \quad \perp \sqsubseteq a$

Finally, the theory $\mathcal{P}^0_\perp$ enjoys the existence of both 0 and $\perp$. In the classical case, the free construction for this theory was investigated in [8]. We call the four theories introduced so far *plain* power theories in contrast to the lower and upper theories which are defined in the sequel.

The *lower* theories $\mathcal{L}$, $\mathcal{L}^0$, $\mathcal{L}_\perp$, and $\mathcal{L}^0_\perp$ are derived from the corresponding plain theories $\mathcal{P}$ through $\mathcal{P}^0_\perp$ by means of the additional inequational axiom $a \sqsubseteq a + b$. The free construction for $\mathcal{L}$ is called *lower* or *Hoare* power construction in the classical case. The *upper* theories $\mathcal{U}$

etc. result from the plain theories by adding the dual axiom $a + b \sqsubseteq b$. The free construction $\mathcal{U}$ is called *upper* or *Smyth* power construction.

In all power theories, there is a commutative, associative, and idempotent operation '+'. The requirement of stability puts severe restrictions on such operations.

Lemma 4.1 Let X be an scpo, and $+ : X \times X \to X$ a stable operation that is commutative and idempotent. Then $a \overline{\uparrow} b$ in X implies $a + b = a \sqcap b$.

Proof: Prop. 2.4 (1) implies $(a, b) \overline{\uparrow} (b, a)$. Then

$a + b \overset{comm.}{=} (a + b) \sqcap (b + a) \overset{stab.}{=} (a \sqcap b) + (b \sqcap a) \overset{id.}{=} a \sqcap b$ □

This lemma implies that addition is strict:

Corollary 4.2 If X is an scpo with least element $\bot$, and $+ : X \times X \to X$ is stable, commutative, and idempotent, then $x + \bot = \bot + x = \bot$ holds for all x in X.

Lemma 4.1 also implies that lower power domains are degenerated.

Proposition 4.3 Non-empty $\mathcal{L}$-algebras have singleton carriers.

Proof: Let a and b be two elements from an $\mathcal{L}$-algebra. By the lower axiom, we get $a \sqsubseteq a + b$ and $b \sqsubseteq a + b$, whence $a \uparrow b$. Lemma 4.1 implies $a + b = a \sqcap b$, and therefore $a = b$.□

The upper and plain power constructions are not degenerated. Consider the scpo $\mathbf{U} = \{1 \sqsubset 0\}$ with addition $0 + 0 = 0$ and $0 + 1 = 1 + 0 = 1 + 1 = 1$. It is easily checked that this addition is stable, commutative, associative, and idempotent. Furthermore, $\mathbf{U}$ is an algebra for all the plain and upper power theories from $\mathcal{P}$ through $\mathcal{U}^0_\bot$. Since $\mathbf{U}$ apparently is non-discrete, we may conclude from Theorem 3.3:

Proposition 4.4 For all separable X, the singleton map $\mathsf{s} : X \to \mathcal{C}X$ is an order embedding, if $\mathcal{C}$ is any of the upper and plain constructions $\mathcal{P}$ through $\mathcal{U}^0_\bot$.

Since separability is implied by continuity (Theorem 7.2), the singleton map is an order embedding for all stable domains that occur in practical semantics.

Problem 2 What about the singleton map if the ground domain is not separable?

Problem 3 Is separability preserved by the power constructions mentioned above?

The next question to consider is the stability of the extension of the various power domain constructions. By Theorem 3.5, we have to check the exponentiability of the power theories. The plain theories $\mathcal{P}$ and $\mathcal{P}^0$ are exponentiable since they are purely equational, and equality of functions is still defined pointwise in the stable case. The theories $\mathcal{P}_\bot$ and $\mathcal{P}^0_\bot$ are also exponentiable because of the following fact:

Proposition 4.5
If $\bot$ is the least element of Y, then $\lambda x.\, \bot$ is the least element of $[X \to Y]$.

From Theorem 3.5, we can now conclude:

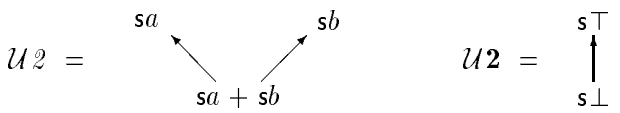
Proposition 4.6
The extension functionals are stable for all plain theories $\mathcal{P}$ through $\mathcal{P}^0_\bot$.

In contrast to this, the upper theories are not exponentiable, and their extension functionals are not stable in general; they are not even monotonic.

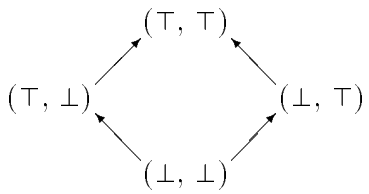
For non-exponentiability, consider the following example: $\mathbf{U} = \{1 \sqsubset 0\}$ is an algebra for all upper theories. We choose $X = \mathbf{2} = \{\bot \sqsubset \top\}$. The function space $[\mathbf{2} \to \mathbf{U}]$ with pointwise addition violates the upper axiom. Representing functions f by pairs $(f\bot, f\top)$, we notice that $(1, 0)$ and $(0, 0)$ are incomparable in the stable order, whence $(1, 0) + (0, 0) \not\sqsubseteq (0, 0)$.

For the theory $\mathcal{U}$, we now present an example showing that the mapping functional M is not always monotonic. For the remaining upper theories, similar examples exist.

Let $\mathcal{2} = \{a, b\}$ be the scpo with two incomparable points a and b, and let $\mathbf{2} = \{\bot \sqsubset \top\}$ as usual. One can show that their $\mathcal{U}$-powerdomains look as follows:



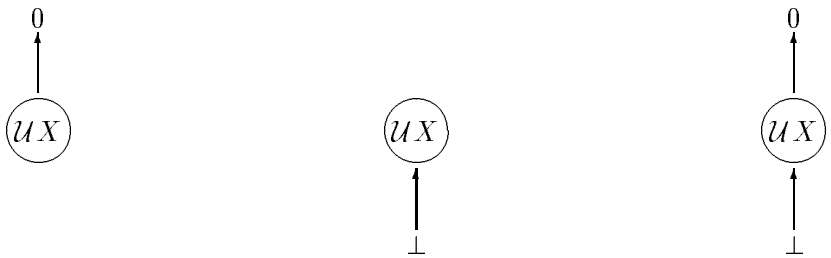
The function space $[\mathcal{2} \to \mathbf{2}]$ with the stable order looks as follows, if f is represented by the pair (fa, fb):



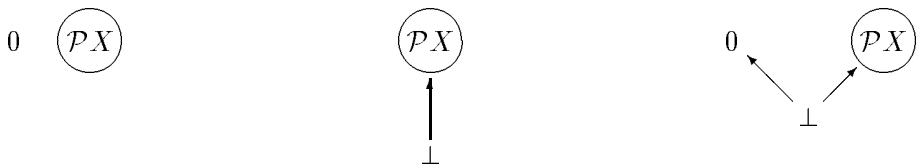
In this domain, $(\top, \bot) \sqsubseteq (\top, \top)$ holds, but $\mathsf{M}(\top, \bot) = (\mathsf{s}a \mapsto \mathsf{s}\top, \mathsf{s}b \mapsto \mathsf{s}\bot, \mathsf{s}a + \mathsf{s}b \mapsto \mathsf{s}\bot)$ is not stably below $\mathsf{M}(\top, \top) = (_ \mapsto \mathsf{s}\top)$.

In the sequel, we show how to derive the upper and plain power domains with zero and/or bottom from the basic power domains $\mathcal{U}X$ and $\mathcal{P}X$.

Theorem 4.7 The power domains $\mathcal{U}^0X, \mathcal{U}_\bot X$, and $\mathcal{U}^0_\bot X$ look as follows:



The power domains $\mathcal{P}^0X$, $\mathcal{P}_\bot X$, and $\mathcal{P}^0_\bot X$ look as follows:



Proof: In the upper theories, 0 is at the top because the upper axiom $a + b \sqsubseteq b$ implies $a = a + 0 \sqsubseteq 0$. Addition can be defined for the extended power domains by means of the rules $a + 0 = 0 + a = a$ and $a + \bot = \bot + a = \bot$. The rule for $\bot$ follows from Cor. 4.2. The verification of the required properties is straightforward by case analysis. □

In the classical case, the power domains with 0 and/or $\bot$ also look as depicted above — except for $\mathcal{P}^0_\bot X$ that has many more elements since $a + \bot$ is different from $\bot$ in general. The structure of the classical $\mathcal{P}^0_\bot X$ was analyzed in [8].

Since the lower constructions are degenerated, and the upper constructions suffer from their non-monotonic extension, only the plain constructions are worth for further investigation. In view of Theorem 4.7, it suffices to consider the stable Plotkin construction $\mathcal{P}$.

Whereas it requires much work to derive an explicit description of $\mathcal{P}X$ for continuous scpo's X, there is a special case of particularly simple scpo's, where an explicit construction is immediate. An scpo X is *fully compatible* iff all pairs of points of X are weakly compatible. In particular, all scpo's with a greatest element are fully compatible.

Proposition 4.8 If X is a fully compatible scpo, then $\mathcal{P}X$ is isomorphic to X.

Proof: Since all pairs of points are weakly compatible, all binary meets exist by Prop. 2.3, and '$\sqcap$' is a total continuous operation, whose stability is immediate. Thus, X itself becomes a stable semilattice by means of '$\sqcap$'. If Y is another stable semilattice and $f : X \to Y$ is stable, then f itself is additive because of $f(a \sqcap b) = fa \sqcap fb = fa + fb$ by Lemma 4.1. Thus, X itself satisfies the universal property of $\mathcal{P}X$ with $\mathsf{s} = id$ and $\mathsf{E}f = f$. □

By Prop. 4.4, the singleton operation $\mathsf{s} : X \to \mathcal{P}X$ is an order embedding for separable scpo's X. The proposition above shows that s is also an order embedding for the non-separable scpo **RC** of subsection 2.4, since **RC** has a greatest element. This leads to the question:

Problem 4 Is $\mathsf{s} : X \to \mathcal{P}X$ an order embedding for all scpo's X?

5 Stable Topology

In preparing the development of an explicit description of $\mathcal{P}X$ for all continuous scpo's X, we now introduce *stable open sets* and their properties. Although they fail to form a topology in the ordinary sense, they allow the application of topological methods to scpo's.

In the classical case, the function space $[X \to \mathbf{2}]$ of first order predicates is isomorphic to ΩX, the set of Scott open subsets of X ordered by inclusion. We want to derive an analogous result for the stable case.

Definition 5.1

(1) A subset O of an scpo X is *stable open* iff it is Scott open, and closed under compatible meets, i.e., for all x, y in O that are compatible (w.r.t. X) their meet $x \sqcap y$ is in O again.

(2) A subset O of an scpo X is *filtered open* iff it is Scott open and filtered, i.e., O is not empty, and every two elements of O have a common lower bound in O.

The importance of these notions is apparent from the following theorem:

Proposition 5.2 For a subset O of an scpo X, the following statements are equivalent:

(1) The function $\chi_O : X \to \mathbf{2}$ with $\chi_O\, x = \top$ iff $x \in O$ is stable.

(2) The set O is stable open.

(3) The set O is a disjoint union of filtered opens, called the *filtered components* of O.

Proof: We only indicate how to prove (2) $\Rightarrow$ (3). Define a binary relation $\sim$ on the stable open set O by $x \sim y$ iff x and y have a common lower bound in O. This is an equivalence relation on O. Its classes are the filtered components of O. □

In the classical case, predicates are ordered pointwise, which corresponds to the inclusion order on Scott open sets. In the stable case, we have to consider the stable order instead.

Proposition 5.3 For two stable open subsets U and V of an scpo X, the following statements are equivalent:

(1) $\chi_U \sqsubseteq \chi_V$ holds in $[X \to \mathbf{2}]$.

(2) $U \subseteq V$ and $V \cap {\downarrow}U \subseteq U$.

(3) There is a stable open subset W of X such that V is the disjoint union of U and V.

Proof: The relation $\chi_U \sqsubseteq \chi_V$ holds iff for $x \sqsubseteq x'$, $\chi_U\, x = \chi_U\, x' \sqcap \chi_V\, x$ holds. The latter equation certainly holds if $\chi_U\, x' = \bot$, whence only the case $\chi_U\, x' = \top$, i.e., $x' \in U$, matters in the following computation.

$$\begin{array}{lll} \chi_U \sqsubseteq \chi_V & \text{iff} & (x \sqsubseteq x' \text{ and } x' \in U) \Rightarrow (x \in U \Leftrightarrow x \in V) \\ & \text{iff} & {\downarrow}U \cap U = {\downarrow}U \cap V \\ & \text{iff} & U \subseteq V \text{ and } V \cap {\downarrow}U \subseteq U \end{array}$$

This proves the equivalence between (1) and (2). The proof that (2) and (3) are equivalent is omitted. □

We call this order the *open order* '$\sqsubseteq_{\mathrm{o}}$'. By part (3), $U \sqsubseteq_{\mathrm{o}} V$ means that V has more, but not larger filtered components than U. The poset of stable open subsets of X ordered by '$\sqsubseteq_{\mathrm{o}}$' is called $\Omega_{\mathrm{s}}X$.

Propositions 5.2 and 5.3 imply:

Proposition 5.4 The function scpo $[X \to \mathbf{2}]$ is isomorphic to the poset $\Omega_{\mathrm{s}}X$.

This proposition enables us to characterize separability by means of stable open sets.

Proposition 5.5 For an scpo X, the following statements are equivalent:

(1) X is separable.

(2) In X, $x \sqsubseteq x'$ holds iff for all stable open sets O, x in O implies x' in O.

Thus, separability is the topological T0 property in the stable setting.

In the stable world, the stable open sets play the same role as the Scott open sets do in the classical world. Unfortunately, their behavior is more complex than that of Scott open sets.

Proposition 5.6 Finite intersections, $\subseteq$-directed unions, and disjoint unions of stable open sets are stable open.

Binary unions of stable open sets are not stable open in general. Thus, the stable open sets do not form a topology on X. Nevertheless, with some care we may apply topological methods. For instance, stable functions are continuous in a topological sense:

Proposition 5.7 If X and Y are scpo's and $f : X \to Y$ is stable, then for every stable open set O of Y, its inverse image $f^{-1}[O]$ is stable open in X.

Equivalence holds iff Y is separable:

Proposition 5.8 Let $f : X \to Y$ be a function from an scpo X to a separable scpo Y. If $f^{-1}[O]$ is stable open in X for every stable open set O of Y, then f is stable.

There is a method to produce new stable opens from given ones, which seems to have no analogue in the classical world.

Proposition 5.9 Let X be an scpo, and A an arbitrary subset of X.

(1) If U is filtered open, then $\uparrow(U \cap \downarrow A)$ is either empty or equals U.

(2) If U is stable open, then $\uparrow(U \cap \downarrow A)$ is stable open, too. It is below U in the open order.

Definition 5.10

The *saturated hull* of a subset A of an scpo X is $\mathsf{sat}\, A = \bigcap\{O \mid O \supseteq A, O \text{ is stable open}\}$. The set A is *saturated* iff $\mathsf{sat}\, A = A$.

In the classical case, $\mathsf{sat}\, A$ would be identical to $\uparrow A$, and saturated sets would be just upper sets. In the stable case however, $\mathsf{sat}\, A$ is much larger than $\uparrow A$ in general. In the scpo of Figure 1 for instance, $\mathsf{sat}\,\{x, y\}$ is the whole scpo.

6 Second Order Predicates

In the classical case, all the known power domain constructions can be described in terms of *second order predicates* if the ground domain is continuous [12]. First order predicates are functions from the ground domain to some domain of logical values, whereas second order predicates are functions from first order predicates to logical values. Intuitively, the second order predicate P associated with a power domain element A tells which first order predicates p are satisfied by some member of A: $Pp = 1$ iff $\exists a \in A : p\, a = 1$.

In the stable case, we shall use the scpo $\mathbf{U} = \{1 \sqsubset 0\}$ as our domain of logical values. It is equipped with a stable disjunction '+' defined by $0 + 0 = 0$ and $0 + 1 = 1 + 0 = 1 + 1 = 1$. Besides this logical notation, we shall also use the isomorphic domain-theoretic notation $\mathbf{2} = \{\bot \sqsubset \top\}$ with operation '$\sqcap$'.

In subsection 6.1, we introduce $\mathcal{P}^2$ with $\mathcal{P}^2 X = [[X \to \mathbf{2}] \to \mathbf{2}]$ and show that it is a functor from scpo's to separable stable semilattices. In subsection 6.2, we present four possible restrictions on second order predicates. In 6.3, they are translated into the language of stable open sets, and shown to be preserved by all power operations.

In 6.4, the restrictions on second order predicates are used to restrict the functor $\mathcal{P}^2$ to a new functor $\mathcal{P}_\mathrm{f}$. Its 'power domains' can be described in terms of second order predicates as well as in terms of open filters of stable open sets. In 6.5, we show the connection between these filters and non-empty saturated *stably compact* sets.

6.1 The Functor $\mathcal{P}^2$

Now, we investigate the spaces of second order predicates $\mathcal{P}^2X = [[X \to \mathbf{U}] \to \mathbf{U}]$. We show that $\mathcal{P}^2$ forms a functor from scpo's to separable stable semilattices.

Proposition 6.1

(1) For every scpo X, $\mathcal{P}^2X$ is a separable stable semilattice with $P + Q = \lambda p.\, Pp + Qp$.

(2) Let $\mathsf{M} : [X \to Y] \to [\mathcal{P}^2X \to \mathcal{P}^2Y]$ be defined by $\mathsf{M}fP = \lambda q^{[Y \to \mathbf{U}]}.\, P(q \circ f)$. Then M is a stable function that maps stable functions to additive stable functions, and $(\mathcal{P}^2, \mathsf{M})$ is a functor.

(3) Let $\mathsf{s}_X : X \to \mathcal{P}^2X$ be defined by $\mathsf{s}x = \lambda p.\, px$. The functions s_X are stable and form a natural transformation.

Proof: Obviously, $\mathcal{P}^2X$ is an scpo. Since $\mathbf{U} \cong \mathbf{2}$ is separable, $\mathcal{P}^2X$ is separable by Theorem 2.6 (3). All the defined functions are stable because they are defined by λ-expressions. The claimed properties can be shown by equational reasoning in the λ-calculus. □

Remember that Prop. 5.4 relates predicates to stable open sets. It also applies to second order predicates.

Proposition 6.2 For every scpo X, $\mathcal{P}^2X$ is isomorphic to $\Omega_{\mathrm{s}}(\Omega_{\mathrm{s}}X)$. The operations in the two representations are given by the following table.

$[[X \to \mathbf{2}] \to \mathbf{2}]$			$\Omega_{\mathrm{s}}(\Omega_{\mathrm{s}}X)$		
$\bigsqcup_{i\in I} P_i$	$=$	$\lambda p.\, \bigsqcup_{i\in I} P_i p$	$\bigsqcup_{i\in I} \mathcal{A}_i$	$=$	$\bigcup_{i\in I} \mathcal{A}_i$
$P + Q$	$=$	$\lambda p.\, Pp + Qp$	$\mathcal{A} + \mathcal{B}$	$=$	$\mathcal{A} \cap \mathcal{B}$
$\mathsf{s}x$	$=$	$\lambda p.\, px$	$\mathsf{s}x$	$=$	$\{U \mid x \in U\}$
$\mathsf{M}fP$	$=$	$\lambda p.\, P\,(p \circ f)$	$\mathsf{M}f\mathcal{A}$	$=$	$\{V \mid f^{-1}[V] \in \mathcal{A}\}$

Here, '$\bigsqcup$' means directed join, and '$\bigcup$' means $\sqsubseteq_{\mathrm{o}}$-directed union.

Proof: The isomorphism $\omega : [[X \to \mathbf{U}] \to \mathbf{U}] \to \Omega_{\mathrm{s}}(\Omega_{\mathrm{s}}X)$ is given by $\omega P = \{U \in \Omega_{\mathrm{s}}X \mid P\chi_U = 0\}$, where $\chi_U x = 0$ iff $x \in U$. (Remember that 0 in $\mathbf{U}$ corresponds to $\top$ in $\mathbf{2}$.) To verify the operations of $\Omega_{\mathrm{s}}(\Omega_{\mathrm{s}}X)$, prove $\omega(\bigsqcup_{i\in I} P_i) = \bigcup_{i\in I} \omega P_i$, $\omega(P + Q) = \omega P \cap \omega Q$, etc.□

6.2 Restrictions on Second Order Predicates

The full set $\mathcal{P}^2X$ of second order predicates contains much junk that cannot be reached by the operations s, '+', M, and directed joins.[2] We now present four restrictions on second order predicates that are preserved by all operations (cf. next subsection).

(1) $P\,(p + q) = Pp + Pq$ (additivity)
Here, addition on first order predicates is defined pointwise: $p + q = \lambda x.\, px + qx$.

(2) $P\,(\lambda x.\, 0) = 0$ (empty case of additivity)

(3) $P\,(\lambda x.\, 1) = 1$ (non-emptiness)

(4) If $(p_i)_{i\in I}$ is directed in the pointwise order, then $P\,(\bigsqcup_{i\in I} p_i) = \bigsqcup_{i\in I}\,(P\,p_i)$ (pointwise continuity)

[2] Compatible meets are covered by '+' because of Lemma 4.1.

The first three restrictions are the same as in the definition of the classical upper power construction $\mathcal{U}_\mathrm{f}$. Restriction (4) is specific for the stable case. As a continuous function, P has to preserve joins of families which are directed in the stable order. Restriction (4) requires it to preserve even joins of families which are directed in the pointwise order.

In Section 6.4 of the report [9], examples are presented showing that the four restrictions are independent from each other if all algebraic scpo's are considered. On the other hand, restriction (4) is implied by restrictions (1) through (3) and continuity of P in case of algebraic scpo's, whose bases do not contain infinitely descending sequences [9, Section 6.6]. All algebraic scpo's with property I and in particular the stable bifinites belong to this class.

One may invent some more sophisticated restrictions, but the existing ones are enough in the continuous case: if X is continuous, every member of $\mathcal{P}^2X$ satisfying the four restrictions can be built using s, '+', and directed joins (see Theorem 7.22).

6.3 Restrictions on Second Order Open Sets

In the previous subsection, we presented four restrictions on second order predicates, and claimed, but not proved, that they are preserved by all operations. In this subsection, we translate these restrictions into the representation of $\mathcal{P}^2X$ as stable open sets of stable open sets. In this representation, the proof of their preservation is straightforward.

In Subsection 6.1, we presented an isomorphism $\omega : \mathcal{P}^2X \to \Omega_\mathrm{s}(\Omega_\mathrm{s}X)$. Remember that $\Omega_\mathrm{s}X$ is the poset of stable opens of X ordered by the stable order '$\sqsubseteq_\mathrm{o}$'. In the following proposition, we also need the poset $\Omega_\mathrm{s}^\mathrm{p}X$ of stable opens of X ordered by set inclusion.

Proposition 6.3 A second-order predicate P in $\mathcal{P}^2X$ satisfies the four restrictions of Subsection 6.2 if and only if ωP is a filtered open in $\Omega_\mathrm{s}^\mathrm{p}X$ which does not contain $\emptyset$.

Proof: Remember $U \in \omega P$ iff $P\chi_U = 0$, and $\chi_U x = 0$ iff $x \in U$.

Restriction (1) is $P(p+q) = Pp + Pq$, or $P(p+q) = 0$ iff $Pp = 0$ and $Pq = 0$. Translated into the language of sets, this means $U \cap V \in \omega P$ iff $U \in \omega P$ and $V \in \omega P$ for stable open sets U and V, i.e., ωP is an upper set in $\Omega_\mathrm{s}^\mathrm{p}X$ and closed under binary intersection.

Restriction (2) is $P(\lambda x.\,0) = 0$. Since $\lambda x.\,0 = \chi_X$, this means $X \in \omega P$.

Restriction (3) is $P(\lambda x.\,1) = 1$. Since $\lambda x.\,1 = \chi_\emptyset$, this means $\emptyset \notin \omega P$.

Restriction (4) refers to pointwise directed joins of predicates, which corresponds to directed joins in $\Omega_\mathrm{s}^\mathrm{p}X$. □

Notice that although the ω-images of the restricted second order predicates are characterized as certain subsets of $\Omega_\mathrm{s}^\mathrm{p}X$, their relative order is inherited from $\Omega_\mathrm{s}(\Omega_\mathrm{s}X)$. Note also that the filtered opens in $\Omega_\mathrm{s}^\mathrm{p}X$ are automatically stable opens in $\Omega_\mathrm{s}X$. Therefore, the latter property does not occur explicitly in Prop. 6.3.

6.4 The Functor $\mathcal{P}_\mathrm{f}$

In the sequel, we shall only consider those second order predicates that satisfy the four restrictions.

Definition 6.4 For every scpo X, let $\mathcal{P}_{\mathrm{f}}X$ be the set of all second order predicates in $\mathcal{P}^2X$ that satisfy the restrictions (1) through (4) of Subsection 6.2.

Second, let $\mathcal{P}_{\Phi}X$ be the set of filtered opens of $\Omega^{\mathrm{p}}_{\mathrm{s}}X$ ordered as members of $\Omega_{\mathrm{s}}(\Omega_{\mathrm{s}}X)$, i.e., by $\mathcal{F} \sqsubseteq \mathcal{G}$ iff $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{G} \cap {\downarrow}\mathcal{F} \subseteq \mathcal{F}$, where the lower closure '$\downarrow$' refers to the $\sqsubseteq_{\mathrm{o}}$-order of stable opens.

By Prop. 6.3, $\mathcal{P}_{\mathrm{f}}X$ and $\mathcal{P}_{\Phi}X$ are isomorphic for all scpo's X. Since all our operations preserve the restrictions, $(\mathcal{P}_{\mathrm{f}}, \mathsf{M})$ also is a functor from scpo's to separable stable semilattices, and $\mathsf{s} : X \to \mathcal{P}_{\mathrm{f}}X$ is a natural transformation. In the $\mathcal{P}_{\Phi}$-representation, the operations are given by $\mathsf{s}x = \{O \mid x \in O\}$, $\mathcal{F} + \mathcal{G} = \mathcal{F} \cap \mathcal{G}$, and $\mathsf{M}f\mathcal{F} = \{O \mid f^{-1}[O] \in \mathcal{F}\}$. Directed joins are given by union.

One can easily show that $\mathsf{s} : X \to \mathcal{P}_{\mathrm{f}}X$ is an order embedding for all separable scpo's X. Conversely, if s is an embedding, then X is separable by Theorem 2.6. Let us consider the non-separable scpo $\mathbf{RC}$ of subsection 2.4. We noted there that the only stable maps from $\mathbf{RC}$ to $\mathbf{2}$ are the two constant maps $\lambda x.0$ and $\lambda x.1$. Their image is prescribed by restrictions (2) and (3). Thus, $\mathcal{P}_{\mathrm{f}}(\mathbf{RC})$ has exactly one element, whereas $\mathbf{RC}$ is uncountable. On the other hand, $\mathcal{P}(\mathbf{RC}) = \mathbf{RC}$ holds by Prop. 4.8.

6.5 Stably Compact Sets

Above, we introduced the power domains $\mathcal{P}_{\Phi}X$ in terms of filtered opens in $\Omega^{\mathrm{p}}_{\mathrm{s}}X$. We now try to reduce these second order sets to first order sets, i.e., to describe them by certain subsets of X.

As the first order description of a filter $\mathcal{F}$ in $\mathcal{P}_{\Phi}X$, we take the intersection $\psi(\mathcal{F}) = \bigcap \mathcal{F}$ of all stable opens contained in $\mathcal{F}$. Conversely, for $A \subseteq X$, let $\varphi A = \{O \in \Omega_{\mathrm{s}}X \mid A \subseteq O\}$.

This definition covers the intuition of existential quantification. O is in φA iff $A \subseteq O$. Translated into the language of predicates, this means $\varphi A p = 0$ iff $p\,a = 0$ for all a in A, or equivalently $\varphi A p = 1$ iff there is a in A with $p\,a = 1$.

Note that $\psi(\varphi A) = \mathsf{sat}\,A$ holds by definition of the saturated hull. Thus, φ becomes injective if we restrict ourselves to saturated sets. $\psi\mathcal{F}$ is saturated for all open filters $\mathcal{F}$.

Next, let us consider when φA is a filtered open in $\Omega^{\mathrm{p}}_{\mathrm{s}}X$. The set φA is a filter for all sets A. Condition $\emptyset \notin \varphi A$ is satisfied iff A is non-empty. Finally, φA is Scott open in $\Omega^{\mathrm{p}}_{\mathrm{s}}X$ iff A has the following property:

Definition 6.5

A subset A of an scpo X is *stably compact* iff for all $\subseteq$-directed families $(O_i)_{i\in I}$ of stable open sets with $A \subseteq \bigcup_{i\in I} O_i$, there is some k in I with $A \subseteq O_k$.

The discussion above indicates that we should consider non-empty saturated stably compact sets.

Definition 6.6 For every scpo X, let $\mathcal{P}_{\mathrm{k}}X$ be the set of all non-empty saturated stably compact subsets of X ordered by $A \sqsubseteq_{\mathrm{k}} B$ iff $A \supseteq B$, and $A \subseteq U$ implies $A \subseteq {\uparrow}(U \cap {\downarrow}B)$ for all stable open sets U.

The seemingly strange 'compact order' $\sqsubseteq_{\mathrm{k}}$ was chosen so that φ becomes an order embedding. By Lemma 7.9, it will be related to the familiar Egli-Milner order.

Proposition 6.7 For every scpo X, the mapping $\varphi : \mathcal{P}_{\mathrm{k}}X \to \mathcal{P}_{\Phi}X$ is an order embedding: $A \sqsubseteq_{\mathrm{k}} B$ iff $\varphi A \sqsubseteq_{\mathrm{o}} \varphi B$.

Proof: Since A and B are saturated, $A \supseteq B$ is equivalent to $\varphi A \subseteq \varphi B$. We have to show that under the condition $A \supseteq B$, the implication $A \subseteq U \Rightarrow A \subseteq \uparrow(U \cap \downarrow B)$ for all stable open sets U is equivalent to $\varphi B \cap \downarrow \varphi A \subseteq \varphi A$.

First assume $A \sqsubseteq_{\mathrm{k}} B$. Let V be in $\varphi B \cap \downarrow \varphi A$. Then $B \subseteq V$ and $V \sqsubseteq_{\mathrm{o}} U$ for some U with $A \subseteq U$. By hypothesis, the latter inclusion implies $A \subseteq \uparrow(U \cap \downarrow B) \subseteq \uparrow(U \cap \downarrow V)$. By $V \sqsubseteq_{\mathrm{o}} U$, $U \cap \downarrow V \subseteq V$ holds, whence $A \subseteq V$, i.e., V is in φA.

Conversely, assume $\varphi B \cap \downarrow \varphi A \subseteq \varphi A$ holds, and let $A \subseteq U$, i.e., U in φA. Let $V = \uparrow(U \cap \downarrow B)$. By Prop. 5.9, V is stable open with $V \sqsubseteq_{\mathrm{o}} U$, whence V in $\downarrow \varphi A$. By $B \subseteq A \subseteq U$, $B \subseteq V$ holds. Thus, $V \in \varphi B \cap \downarrow \varphi A \subseteq \varphi A$, i.e., $A \subseteq V$. □

Later, we need the following property of '$\sqsubseteq_{\mathrm{k}}$':

Proposition 6.8 Let A, B, and C be elements of $\mathcal{P}_{\mathrm{k}}X$ with $A \supseteq B \supseteq C$ and $A \sqsubseteq_{\mathrm{k}} C$. Then $A \sqsubseteq_{\mathrm{k}} B$ follows.

Proof: $A \supseteq B$ holds by hypothesis. If $A \subseteq U$ for some stable open set U, then $A \sqsubseteq_{\mathrm{k}} C$ implies $A \subseteq \uparrow(U \cap \downarrow C) \subseteq \uparrow(U \cap \downarrow B)$, where the last inclusion follows from $B \supseteq C$. □

In view of Prop. 6.7, we are particularly interested in the case where φ is surjective. We call this case *stably sober* in analogy to the classical notion of sobriety.

Definition 6.9 An scpo X is *stably sober* iff $\varphi : \mathcal{P}_{\mathrm{k}}X \to \mathcal{P}_{\Phi}X$ is surjective.

Surjective order embeddings are order isomorphisms, whence we may conclude:

Proposition 6.10 If X is stably sober, then $\mathcal{P}_{\mathrm{k}}X$ is an scpo isomorphic to $\mathcal{P}_{\Phi}X$. The operations '$+$', s, and M of $\mathcal{P}_{\Phi}$ translate into the following operations for $\mathcal{P}_{\mathrm{k}}$:
$A + B = \mathsf{sat}\,(A \cup B)$, $\mathsf{s}x = \mathsf{sat}\,\{x\}$, and $\mathsf{M}fA = \mathsf{sat}\,f[A]$.
If X is separable, then $\mathsf{sat}\,\{x\} = \uparrow\{x\}$.

Proof: It is not difficult to check that $A + B$, $\mathsf{s}x$, and $\mathsf{M}fA$ are back in $\mathcal{P}_{\mathrm{k}}X$ again.

A stable open set O is in $\varphi(A + B)$ iff $\mathsf{sat}\,(A \cup B) \subseteq O$ iff $A \cup B \subseteq O$ iff $A \subseteq O$ and $B \subseteq O$. Thus, $\varphi(A + B) = \varphi A \cap \varphi B = \varphi A + \varphi B$ holds.

For stable open sets O, $\mathsf{sat}\,\{x\} \subseteq O$ holds iff $x \in O$, whence $\varphi(\mathsf{s}_{\mathrm{k}}\,x) = \{O \mid x \in O\} = \mathsf{s}_{\Phi}\,x$. For separable X, $\mathsf{sat}\,\{x\} = \uparrow\{x\}$ holds by Prop. 5.5.

Finally, $\mathsf{sat}\,f[A] \subseteq O$ iff $f[A] \subseteq O$ iff $A \subseteq f^{-1}[O]$, whence $\varphi(\mathsf{M}_{\mathrm{k}}fA) = \mathsf{M}_{\Phi}f(\varphi A)$ holds. □

It is difficult to judge which scpo's are stably sober. In the next section, we shall see that at least all continuous scpo's are. For more general scpo's, the question is open.

Problem 5 Is there any (any separable) scpo which is not stably sober?

7 The Continuous Case

In this section, we consider continuous scpo's X. For such X, the power domains $\mathcal{P}X$, $\mathcal{P}_{\mathrm{k}}X$, and $\mathcal{P}_{\Phi}X$ are isomorphic and continuous again.

Before these results can be proved, we need several auxiliary notions and lemmas. These are presented in the subsections 7.1 and 7.2. Then, the isomorphism of $\mathcal{P}_{\mathrm{k}}X$ and $\mathcal{P}_{\Phi}X$ for continuous X is shown in 7.3. In 7.4, we show how a (countable) basis of $\mathcal{P}_{\mathrm{k}}X$ can be constructed from a (countable) basis of X, and conclude that the functor $\mathcal{P}_{\mathrm{k}}$ preserves (ω-)continuity. In 7.5, we prove several statements concerning $\mathcal{P}_{\mathrm{k}}X$, e.g., that all elements of $\mathcal{P}_{\mathrm{k}}X$ can be reached from singletons by addition and directed joins, or that for saturated sets, stable compactness and Scott compactness coincide. Finally, we show in 7.6 that $\mathcal{P}_{\mathrm{k}}X$ is a free stable semilattice over X, whence it forms an explicit representation of $\mathcal{P}X$.

7.1 Continuous Scpo's

We now briefly look at continuous scpo's, i.e., those scpo's, whose underlying dcpo is continuous in the usual sense. They enjoy the following important property:

Proposition 7.1 Let X be a continuous scpo. For every point x in a Scott open set U, there exist a filtered open set V and a point y such that $x \in V \subseteq \uparrow\{y\} \subseteq U$ holds.

Proof: Since x is a directed join of points that are way-below x, there is some y in U with $y \ll x$. Thus, $x \in \uparrow\{y\} \subseteq U$ holds. By Exercise I, 3.31 of [5], there is an filtered open V of X with $x \in V \subseteq \uparrow\{y\}$. □

Now, we show that every continuous scpo is separable, as announced in Subsection 2.4.

Theorem 7.2 Every continuous scpo is separable.

Proof: Let $a \not\sqsubseteq b$. Let U be the complement of $\downarrow\{b\}$. Then a is in the Scott open set U, and b is not. By Prop. 7.1, there is a stable open set V such that $a \in V \subseteq U$, whence $b \notin V$.□

In the case of continuous scpo's, the weak compatibility relation of Subsection 2.3 can be characterized neatly:

Proposition 7.3 Let x and y be two points of a continuous scpo. Then x and y are weakly compatible, iff all Scott open sets U and V with $x \in U$ and $y \in V$ meet each other, iff all filtered open sets U and V with $x \in U$ and $y \in V$ meet each other.

Proof: The first equivalence holds since for continuous domains, the Scott topology of a binary product coincides with the product of the Scott topologies. The second equivalence follows from Prop. 7.1. □

7.2 Independent and Strongly Independent Sets

We now introduce independent and strongly independent sets. Finite strongly independent sets will play an important rule in the sequel.

Definition 7.4 A subset A of an scpo is *independent* iff its points are pairwise incompatible, i.e., if u, v in A, then $u \uparrow v$ implies $u = v$. The set A is *strongly independent* iff the same holds with '$\uparrow$' replaced by '$\overline{\uparrow}$'.

Compatible points are weakly compatible, whence strongly independent sets are independent. Thus, these names make sense. In the sequel, we show some properties of finite strongly independent sets.

Lemma 7.5 Let E be a finite strongly independent set in a continuous scpo. Every e in E can be associated with a filtered open set O_e such that $e \in O_e$ holds for all e in E, and O_a and O_b are disjoint for every pair a, b of distinct points of E.

Proof: Let $E = \{e_1, \ldots, e_n\}$. For $i < j$, points e_i and e_j are not weakly dependent, whence by Prop. 7.3, there are disjoint Scott open sets U_{ij} and U_{ji} with e_i in U_{ij} and e_j in U_{ji}. Next, let $V_i = \bigcap_{j \neq i} U_{ij}$ for all i with $1 \leq i \leq n$. For every i, e_i is in V_i. By Prop. 7.1, there are filtered opens O_i with $e_i \subseteq O_i \subseteq V_i$. If $i < j$, then $O_i \subseteq U_{ij}$ and $O_j \subseteq U_{ji}$, whence they are disjoint. □

Proposition 7.6 If X is continuous, then for every finite strongly independent subset E of X, $\uparrow E$ is saturated, i.e., $\mathsf{sat}\, E$ equals $\uparrow E$.

Proof: From Lemma 7.5, we get filtered open sets O_e for every e in E such that O_a and O_b are disjoint for distinct points a and b of E.

We have to show $\mathsf{sat}\, E \subseteq \uparrow E$. Let x be a point that is not in $\uparrow E$. Then, for every e in E, $x \not\sqsupseteq e$ holds. By separability (Theorem 7.2) and Prop. 5.5, there are stable open sets U_e with e in U_e and x not in U_e for every e in E. Let $O = \bigcup_{e \in E}(O_e \cap U_e)$. By Prop. 5.6, O is stable open as a disjoint union of stable open sets. Because of $E \subseteq O$ and $x \notin O$, x is not in $\mathsf{sat}\, E$.□

Generally, there are different finite sets with the same saturated hull. A unique representation is obtained by requiring strong independence.

Proposition 7.7 For every non-empty finite set E, there is a *unique* non-empty finite strongly independent set F with $\mathsf{sat}\, E = \mathsf{sat}\, F$ $(= \uparrow F)$.

Proof: As long as E is not strongly independent, choose two distinct weakly compatible points a and b of E and replace them by their meet $a \sqcap b$, which exists by Prop. 2.3. Doing so does not change the saturated hull, since by Prop. 2.4 (3) and the correspondence between stable opens and stable maps to $\mathbf{2}$, for every stable open set O, $a \sqcap b$ is in O iff both a and b are in O. Every such transformation step reduces the size of E by 1. By finiteness, this procedure will eventually stop yielding a non-empty finite strongly independent set F.

For strongly independent finite F, $\mathsf{sat}\, F = \uparrow F$ holds by Prop. 7.6. This implies uniqueness: strongly independent sets are antichains, and for two antichains F_1 and F_2, $\uparrow F_1 = \uparrow F_2$ implies $F_1 = F_2$. □

Non-empty finite sets generate members of $\mathcal{P}_{\mathrm{k}} X$:

Proposition 7.8 If E is a non-empty finite set, then $\mathsf{sat}\, E$ is in $\mathcal{P}_{\mathrm{k}} X$.

If strongly independent sets are involved, then the strange compact order can be reduced to the familiar Egli-Milner order.

Lemma 7.9

(1) Let A and B be two non-empty sets such that $\mathsf{sat}\, A$ and $\mathsf{sat}\, B$ are stably compact. If $A \sqsubseteq_{\mathrm{EM}} B$, i.e., $\uparrow A \supseteq B$ and $A \subseteq \downarrow B$, then $\mathsf{sat}\, A \sqsubseteq_{\mathrm{k}} \mathsf{sat}\, B$ follows.

(2) If the set A of (1) is finite and strongly independent, then $A \sqsubseteq_{\mathrm{EM}} B$ and $\mathsf{sat}\, A \sqsubseteq_{\mathrm{k}} \mathsf{sat}\, B$ are equivalent.

Proof: For this proof, let $S = \mathsf{sat}\, B$. The inclusion $\uparrow A \supseteq B$ implies $\mathsf{sat}\, A = \mathsf{sat} \uparrow A \supseteq \mathsf{sat}\, B$. To show $\mathsf{sat}\, A \sqsubseteq_{\mathrm{k}} S$, let $\mathsf{sat}\, A \subseteq O$, where O is stable open. From $A \subseteq \downarrow B$, we conclude $A \subseteq \mathsf{sat}\, A \cap \downarrow B \subseteq O \cap \downarrow S$, whence $\mathsf{sat}\, A \subseteq \uparrow(O \cap \downarrow S)$ as required ($\uparrow(O \cap \downarrow S)$ is stable open by Prop. 5.9).

For (2), assume $\mathsf{sat}\, A \sqsubseteq_{\mathrm{k}} S$. By Prop. 7.6, $\mathsf{sat}\, A = \uparrow A$ holds. Thus, $\uparrow A \supseteq S \supseteq B$ immediately follows. Assume A were not a subset of $\downarrow B$. Then there is x in A with x not in $\downarrow B$. Let $A' = A \setminus \{x\}$. From Lemma 7.5, we obtain two disjoint stable open sets O_x and O' with $x \in O_x$ and $A' \subseteq O'$. (O' is the disjoint union of the sets O_y for y in A'.) From $B \subseteq \uparrow A$ and $x \notin \downarrow B$, $B \subseteq \uparrow A' \subseteq O'$ follows, whence $S \subseteq O'$. Thus, O_x and S are disjoint, and since O_x is upper, O_x and $\downarrow S$ are disjoint as well. By $\uparrow A \sqsubseteq_{\mathrm{k}} S$, the inclusion $\uparrow A \subseteq O_x \cup O'$ implies $\uparrow A \subseteq \uparrow((O_x \cup O') \cap \downarrow S) = \uparrow(O' \cap \downarrow S) \subseteq O'$, whence $x \in O'$ in contradiction to $x \in O_x$ and $O_x \cap O' = \emptyset$. □

Proposition 7.10 Let A be independent. Then $A \sqsubseteq_{\mathrm{EM}} B$ holds iff there is a surjective function $\alpha : B \to A$ with $b \sqsupseteq \alpha(b)$ for all b in B.

7.3 Stable Sobriety of Continuous Scpo's

Let us now consider a fixed continuous scpo X, and fix a basis $\mathcal{B}$ of X. Let $\mathcal{B}^*$ be the set of all $\uparrow E$ where E is a non-empty finite strongly independent subset of $\mathcal{B}$. By the results of the previous subsection, $\mathcal{B}^*$ is a subset of $\mathcal{P}_{\mathrm{k}}X$. We shall soon prove that it is even a basis of $\mathcal{P}_{\mathrm{k}}X$, but before this, we want to show that X is stably sober, i.e., $\mathcal{P}_{\mathrm{k}}X$ and $\mathcal{P}_{\Phi}X$ are isomorphic. To obtain this result, we have to approximate the stable open sets of X.

Lemma 7.11 Let X be a continuous scpo with basis $\mathcal{B}$.

(1) Given a filtered open set O, let $\mathcal{D}$ be the set of all filtered open sets U such that there is an element b of $\mathcal{B}$ with $U \subseteq \uparrow\{b\} \subseteq O$. Then $\mathcal{D}$ is $\subseteq$-directed, and its union is O.

(2) For every non-empty stable open set O, there is a set $\mathcal{D}$ of stable open sets such that for every U in $\mathcal{D}$, there is some B in $\mathcal{B}^*$ with $U \subseteq B \subseteq O$, and $\mathcal{D}$ is $\subseteq$-directed with union O.

Proof: For (1), use the fact that O is filtered together with Prop. 7.1. For (2), let $O = \bigcup_{i \in I} O_i$ be the partition of O into disjoint filtered components. For every O_i, let $\mathcal{D}_i$ be the directed set according to (1). Let $\mathcal{D}$ consist of all $\bigcup_{j \in J} U_j$, where J is a finite subset of I, and $U_j \in \mathcal{D}_j$ for all j in J. □

In the proof of stable sobriety, we also need a version of Rudin's Lemma:

Lemma 7.12 Let X be a dcpo, and let $\mathcal{E}$ be a set of finite subsets of X such that $\{\uparrow E \mid E \in \mathcal{E}\}$ is $\supseteq$-directed. If $\bigcap_{E \in \mathcal{E}} \uparrow E \subseteq O$ for some Scott open set O, then there is some E in $\mathcal{E}$ with $\uparrow E \subseteq O$.

Proof: Our version of Rudin's Lemma is derived from Jung's version, which is Theorem 4.11 in [15]. The derivation of our version can be found in [7, 10]. □

With these lemmas, we can now state the decisive property:

Proposition 7.13 Let $\mathcal{F}$ be a filtered open in $\Omega^{\mathrm{p}}_{\mathrm{s}}X$, where X is a continuous scpo. Then $\bigcap \mathcal{F}$ is non-empty, saturated, and stably compact, and $\varphi(\bigcap \mathcal{F}) = \mathcal{F}$ holds.

Proof: Since $\mathcal{F}$ is Scott open, Lemma 7.11 implies that for every U in $\mathcal{F}$, there exist a non-empty finite strongly independent set E and a stable open set V with $U \supseteq \uparrow E \supseteq V$ and V in $\mathcal{F}$.

Let $\mathcal{E}$ be the set of all non-empty finite strongly independent sets E with $\uparrow E \supseteq V$ for some V in $\mathcal{F}$. By the property of the previous paragraph, $\mathcal{E}' = \{\uparrow E \mid E \in \mathcal{E}\}$ is $\supseteq$-directed with $\bigcap \mathcal{E}' = \bigcap \mathcal{F}$.

If O is in $\mathcal{F}$, then $\bigcap \mathcal{F} \subseteq O$. Thus, $\mathcal{F} \subseteq \varphi(\bigcap \mathcal{F})$ holds. Conversely, if O is in $\varphi(\bigcap \mathcal{F})$, then $\bigcap \mathcal{E}' = \bigcap \mathcal{F} \subseteq O$. By Rudin's Lemma 7.12, there is E in $\mathcal{E}$ with $\uparrow E \subseteq O$. By definition of $\mathcal{E}$, there is O' in $\mathcal{F}$ with $O' \subseteq \uparrow E$, whence O in $\mathcal{F}$ follows.

Thus, we know $\varphi(\bigcap \mathcal{F}) = \mathcal{F}$. This equality implies that $\bigcap \mathcal{F}$ is non-empty and stably compact. It is saturated as an intersection of stable open sets. □

Now, we can formulate:

Theorem 7.14 If X is a continuous scpo, then $\mathcal{P}_{\mathrm{k}}X$ and $\mathcal{P}_{\Phi}X$ are isomorphic via φ. Hence, $\mathcal{P}_{\mathrm{k}}X$ is an scpo where directed joins are given by intersection.

Proof: By Prop. 6.7, φ is an order embedding, and by Prop. 7.13, it is surjective. Thus, it is an order isomorphism.

For the directed joins, we have to do a bit more. Let $(K_i)_{i\in I}$ be a $\sqsubseteq_{\mathrm{k}}$-directed family of members of $\mathcal{P}_{\mathrm{k}}X$. We have to show $\bigcap_{i\in I} K_i \in \mathcal{P}_{\mathrm{k}}X$, and $\varphi(\bigcap_{i\in I} K_i) = \bigcup_{i\in I} \varphi K_i$, since directed joins in $\mathcal{P}_{\Phi}X$ are given by union. By Prop. 7.13, it suffices to show $\bigcap_{i\in I} K_i = \bigcap \bigcup_{i\in I} \varphi K_i$. By set theory, $\bigcap \bigcup_{i\in I} \varphi K_i$ equals $\bigcap_{i\in I} \bigcap \varphi K_i$. This set equals $\bigcap_{i\in I} K_i$ because $\bigcap \varphi K_i = \mathsf{sat}\, K_i = K_i$. □

7.4 Continuity of $\mathcal{P}_{\mathrm{k}}X$

In this subsection, we shall prove that whenever X is a continuous scpo with basis $\mathcal{B}$, then $\mathcal{P}_{\mathrm{k}}X$ is continuous with basis $\mathcal{B}^*$. We need an auxiliary relation in this proof.

Definition 7.15 For A and B in $\mathcal{P}_{\mathrm{k}}X$, let $A \prec_{\mathrm{k}} B$ iff $A \sqsubseteq_{\mathrm{k}} B$ and there is some stable open O with $A \supseteq O \supseteq B$.

The significance of this relation lies in the following fact:

Proposition 7.16 If $A \prec_{\mathrm{k}} B$, then $A \ll B$.

Proof: Let $B \sqsubseteq_{\mathrm{k}} \bigsqcup_{i\in I} C_i$ for some directed family $(C_i)_{i\in I}$ of $\mathcal{P}_{\mathrm{k}}X$. Then $\bigsqcup_{i\in I} C_i \subseteq B \subseteq O$ follows, whence O is in $\varphi(\bigsqcup_{i\in I} C_i) = \bigcup_{i\in I} \varphi C_i$ (cf. proof of Theorem 7.14). Thus, there is some k in I with $C_k \subseteq O$. From $A \supseteq O \supseteq C_k \supseteq \bigsqcup_{i\in I} C_i$ and $A \sqsubseteq_{\mathrm{k}} B \sqsubseteq_{\mathrm{k}} \bigsqcup_{i\in I} C_i$, the relation $A \sqsubseteq_{\mathrm{k}} C_k$ follows by Prop. 6.8. □

Next, we show that there are enough approximants.

Proposition 7.17 Let X be a continuous scpo with basis $\mathcal{B}$, K a non-empty stably compact subset and U a stable open subset of X. If $K \subseteq U$, then there exists a member B of $\mathcal{B}^*$ with $B \prec_{\mathrm{k}} K$ and $B \subseteq U$.

Proof: By combination of Lemma 7.11 with the definition of stable compactness, we immediately obtain a stable open set V and a non-empty finite strongly independent subset E of $\mathcal{B}$ with $K \subseteq V \subseteq \uparrow E \subseteq U$. To achieve $\uparrow E \prec_{\mathrm{k}} K$, we have to change V and E.

Let $V' = \uparrow(V \cap \downarrow K)$, which is stable open by Prop. 5.9, and $E' = E \cap \downarrow K$, which is finite and strongly independent again. First, $K \subseteq V$ and $K \subseteq \downarrow K$ implies $K \subseteq V'$. Second, $V \subseteq \uparrow E$ implies $V' \subseteq \uparrow(\uparrow E \cap \downarrow K)$. The inclusion $\uparrow(\uparrow E \cap \downarrow K) \subseteq \uparrow(E \cap \downarrow K) = \uparrow E'$ is easily verified. Third, $\uparrow E' \subseteq \uparrow E \subseteq U$ holds. Thus, we obtain the chain of inclusions $K \subseteq V' \subseteq \uparrow E' \subseteq U$. From $\uparrow E' \supseteq K$ and $E' \subseteq \downarrow K$, $\uparrow E' \sqsubseteq_{\mathrm{k}} K$ follows by Lemma 7.9. □

With the facts collected so far, we can prove:

Proposition 7.18 Let X be a continuous scpo with basis $\mathcal{B}$. For every K in $\mathcal{P}_{\mathrm{k}}X$, the set of all B in $\mathcal{B}^*$ with $B \prec_{\mathrm{k}} K$ is $\sqsubseteq_{\mathrm{k}}$-directed with join K.

Proof: Let $\mathcal{D}$ be the set under consideration. Non-emptiness of $\mathcal{D}$ is shown by applying Prop. 7.17 to the situation $K \subseteq X$. For directedness, let A, B in $\mathcal{D}$. Then $A, B \sqsubseteq_{\mathrm{k}} K$ and $K \subseteq U \subseteq A$ and $K \subseteq V \subseteq B$ for some stable open sets U and V. Applying Prop. 7.17 to the situation $K \subseteq U \cap V$, we obtain C in $\mathcal{B}^*$ with $C \subseteq U \cap V$ and $C \prec_{\mathrm{k}} K$. From $A, B \supseteq C \supseteq K$ and $A, B \sqsubseteq_{\mathrm{k}} K$, the relations $A, B \sqsubseteq_{\mathrm{k}} C$ follow by Prop. 6.8.

Finally, we have to show $\bigcap \mathcal{D} = K$. The inclusion $K \subseteq \bigcap \mathcal{D}$ holds by definition of $\mathcal{D}$, and $\bigcap \mathcal{D} \subseteq \bigcap \varphi K = \mathsf{sat}\, K = K$ holds by Prop. 7.17. □

This proposition proves:

Theorem 7.19 If X is an (ω-)continuous scpo with basis $\mathcal{B}$, then $\mathcal{P}_{\mathrm{k}}X$ is an (ω-)continuous scpo with basis $\mathcal{B}^*$.

It is easy to show that the way-below relation of $\mathcal{P}_{\mathrm{k}}X$ coincides with the relation '$\prec_{\mathrm{k}}$' of Def. 7.15.

7.5 Further Properties of $\mathcal{P}_{\mathrm{k}}X$

First, we present a simple condition which implies the way-below property, but is not equivalent to it.

Proposition 7.20 Let X be a continuous scpo. If I is a non-empty finite index set, and $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ are two families of points of X such that $\{b_i \mid i \in I\}$ is strongly independent and $a_i \ll b_i$ holds for all i in I, then $\mathsf{sat}\,\{a_i \mid i \in I\} \ll \mathsf{sat}\,\{b_i \mid i \in I\}$ holds in $\mathcal{P}_{\mathrm{k}}X$.

Proof: Let $A = \{a_i \mid i \in I\}$ and $B = \{b_i \mid i \in I\}$. We show $\mathsf{sat}\, A \prec_{\mathrm{k}} \mathsf{sat}\, B$ and apply Prop. 7.16. First, $A \sqsubseteq_{\mathrm{EM}} B$ holds, whence $\mathsf{sat}\, A \sqsubseteq_{\mathrm{k}} \mathsf{sat}\, B$ by Prop. 7.9 (1). We have to find a stable open set O with $\mathsf{sat}\, A \supseteq O \supseteq \mathsf{sat}\, B$.

By Lemma 7.5, there are pairwise disjoint filtered opens U_i with $b_i \in U_i$ for all i in I. Let $V_i = \{x \in X \mid a_i \ll x\}$. These sets are Scott open, and $b_i \in U_i \cap V_i$ holds. By Prop. 7.1, we get filtered opens W_i such that $b_i \in W_i \subseteq U_i \cap V_i$. The disjoint union $\bigcup_{i \in I} W_i$ is the desired stable open set. □

Besides being continuous scpo's, the power domains $\mathcal{P}_{\mathrm{k}}X$ are stable semilattices. We now investigate how this algebraic structure relates to the domain-theoretic structure.

Proposition 7.21 If the basis $\mathcal{B}$ of X is closed under weakly compatible meets, then the basis $\mathcal{B}^*$ of $\mathcal{P}_{\mathrm{k}}X$ is closed under addition.

Proof: Let A and B in $\mathcal{B}^*$, i.e., $A = \mathsf{sat}\, F$ and $B = \mathsf{sat}\, G$, where F and G are non-empty finite strongly independent subsets of $\mathcal{B}$. Then $A + B = \mathsf{sat}\,(A \cup B) = \mathsf{sat}\,(F \cup G)$, since sat is a closure operator. By Prop. 7.7, there is a unique non-empty finite strongly independent set H with $\mathsf{sat}\,(F \cup G) = \mathsf{sat}\, H$. As can be seen in the proof of Prop. 7.7, the elements of H result from the elements of $F \cup G$ by meets of weakly compatible points. Therefore, they are in $\mathcal{B}$. □

Next, we show that the power domains $\mathcal{P}_{\mathrm{k}}X$ do not contain any junk.

Theorem 7.22 If X is a continuous scpo, then $\mathcal{P}_{\mathrm{k}}X$ cannot be restricted further: all elements can be reached from singletons by addition and directed join.

Proof: By Prop. 7.18, all members of $\mathcal{P}_{\mathrm{k}}X$ are directed joins of elements $\uparrow E$, where E is a non-empty finite strongly independent set. For every such E, $\uparrow E = \sum_{e \in E} \mathsf{s}e$ holds. □

Finally, we prove that in the characterization of $\mathcal{P}_{\mathrm{k}}X$, stable compactness may be replaced by the more familiar Scott compactness.

Theorem 7.23 Let X be a continuous scpo. A saturated subset A of X is stably compact iff it is Scott compact.

Proof: Since stable open sets are special Scott open sets, Scott compact sets are stably compact. For the opposite direction, let A be saturated stably compact with $A \subseteq \bigcup_{i \in I} O_i$, where the sets O_i are Scott open. By Prop. 7.18, there is a $\sqsubseteq_{\mathrm{k}}$-directed set of sets $\uparrow E$ with finite E, whose join is A. Hence, there is a $\supseteq$-directed set of sets $\uparrow E$ with finite E, whose intersection is A. By Rudin's Lemma 7.12, $A \subseteq \uparrow E \subseteq \bigcup_{i \in I} O_i$ holds for some finite E. Thus, there is a finite subset J of the index set I with $A \subseteq \uparrow E \subseteq \bigcup_{j \in J} O_j$. □

7.6 Freeness of $\mathcal{P}_{\mathrm{k}}X$

In this section, we show that for continuous X, $\mathcal{P}_{\mathrm{k}}X$ and $\mathcal{P}X$ are isomorphic, i.e., $\mathcal{P}_{\mathrm{k}}X$ is the free stable semilattice over X.

Let X be a continuous scpo, S a stable semilattice, and $f : X \to S$ a stable map. We have to show that there is a unique additive stable map $\overline{f} : \mathcal{P}_{\mathrm{k}}X \to S$ with $\overline{f} \circ \mathsf{s} = f$. Uniqueness directly follows from Theorem 7.22. The problem is to show the existence of $\overline{f}$.

From the given function f, we can define $f^*E = \sum_{e \in E} fe$ for all non-empty finite subsets E of X. This function obviously satisfies a kind of additivity: $f^*(E \cup E') = f^*E + f^*E'$.

We are not so much interested in finite sets as in their saturated hulls. Thus, we prove:

Proposition 7.24 If $\mathsf{sat}\, E = \mathsf{sat}\, E'$, then $f^*E = f^*E'$.

Proof: By Prop. 7.7 and its proof, there is a unique non-empty finite strongly independent set E'', which can be reached from both E and E' by a finite sequence of transformation steps. Every step consists in replacing two distinct weakly compatible points a and b by their meet $a \sqcap b$. By Prop. 2.4 (3), $f(a \sqcap b) = fa \sqcap fb$ holds. By Prop. 2.4 (2), fa and fb are weakly compatible, whence by Prop. 4.1, $fa \sqcap fb$ equals $fa + fb$. Thus, $f(a \sqcap b) = fa + fb$ holds, whence a transformation step applied to some set does not change its f^*-value. □

By the proposition above, we can safely define: $\widehat{f}(\mathsf{sat}\, E) = f^*E$ for non-empty finite sets E. The function $\widehat{f}$ is defined for all members of X^*, the basis of $\mathcal{P}_{\mathrm{k}}X$ which results from the basis X of X.

Proposition 7.25 $\widehat{f} : X^* \to S$ is additive and monotonic.

Proof: Let $A = \mathsf{sat}\, G$ and $B = \mathsf{sat}\, H$, where G and H are non-empty finite strongly independent sets. Then $\widehat{f}(A + B) = \widehat{f}(\mathsf{sat}\,(G \cup H)) = f^*(G \cup H) = f^*G + f^*H = \widehat{f}A + \widehat{f}B$. If $A \sqsubseteq_{\mathrm{k}} B$, then $G \sqsubseteq_{\mathrm{EM}} H$ by Lemma 7.9. By Prop. 7.10, there is a surjective map $\gamma : H \to G$ with $h \sqsupseteq \gamma h$ for all h in H. Then $\widehat{f}A = f^*G = \sum_{g \in G} fg$, which by idempotence of '+' and surjectivity of γ equals $\sum_{h \in H} f(\gamma h)$, which by monotonicity of f and '+' is below $\sum_{h \in H} fh = f^*H = \widehat{f}B$. □

By Prop. 7.18, every member K of $\mathcal{P}_{\mathrm{k}}X$ is the join of the directed set $\Downarrow K = \{B \in X^* \mid B \ll K\}$. By monotonicity of $\widehat{f}$, the image $\widehat{f}\,[\Downarrow K]$ is directed in S, whence we may define $\overline{f}K = \bigsqcup \widehat{f}\,[\Downarrow K]$. We shall show that this is the desired additive stable function with $\overline{f} \circ \mathsf{s} = f$.

First, $\overline{f}$ is continuous by the results in Section 2.2.6 of [16]. (The reason is that $\Downarrow(\bigsqcup_{i \in I} K_i) = \bigcup_{i \in I} \Downarrow K_i$ for directed families $(K_i)_{i \in I}$.)

Next, we show that $\overline{f}$ extends $\widehat{f}$, i.e., $\overline{f}(\mathsf{sat}\, E) = \widehat{f}(\mathsf{sat}\, E)$ holds for all non-empty finite strongly independent sets E. The relation '$\sqsubseteq$' is obvious by definition of $\overline{f}$. For the opposite direction, we have to show $\overline{f}(\mathsf{sat}\, E) \sqsupseteq f^*E$. Let $E = \{e_1, \ldots, e_n\}$. Every set $\{e_1', \ldots, e_n'\}$ with $e_i' \ll e_i$ for all i is in $\Downarrow E$ by Prop. 7.20. Thus, $\overline{f}(\mathsf{sat}\, E) \sqsupseteq \bigsqcup\{fe_1' + \cdots + fe_n'\}$ holds. By continuity of X, of '+' in S, and of f, this is a directed join which equals $fe_1 + \cdots + fe_n = f^*E$.

In particular, we can conclude $\overline{f}(\mathsf{s}x) = \widehat{f}(\mathsf{sat}\,\{x\}) = fx$. The function $\overline{f}$ is additive on $\mathcal{P}_{\mathrm{k}}X$ because it is continuous, and its restriction $\widehat{f}$ is additive on the basis X^* of $\mathcal{P}_{\mathrm{k}}X$. Additivity implies stability since compatible meets are sums in $\mathcal{P}_{\mathrm{k}}X$ and S. This completes our proof of the freeness of $\mathcal{P}_{\mathrm{k}}X$.

Theorem 7.26 For every continuous scpo X, $\mathcal{P}_{\mathrm{k}}X$ is the free stable semilattice over X. Thus, $\mathcal{P}_{\mathrm{k}}X$ and $\mathcal{P}X$ are isomorphic for continuous X.

8 The Algebraic Case

In this section, we consider the structure of $\mathcal{P}X$ for algebraic X. A dcpo is algebraic iff it has a basis of isolated points. Such a basis contains all isolated points and is therefore uniquely determined. It is contained in every other basis, and we call it the *canonical basis*.

Theorem 8.1 If X is an (ω-)algebraic scpo with canonical basis $\mathcal{B}$, then $\mathcal{P}_{\mathrm{k}}X$ is (ω-)algebraic with canonical basis $\mathcal{B}^*$ (which is the collection of all sets $\uparrow E$ where E is a non-empty finite independent subset of $\mathcal{B}$).

Proof: Since $\uparrow x$ is a filtered open if x is isolated, sets of isolated points are independent iff they are strongly independent. Thus, the definition of $\mathcal{B}^*$ in the theorem is equivalent to the one used in the previous section. By Theorem 7.19, $\mathcal{B}^*$ is a basis of $\mathcal{P}_{\mathrm{k}}X$. Since x is isolated iff $x \ll x$, the members of $\mathcal{B}^*$ are isolated by Prop. 7.20. □

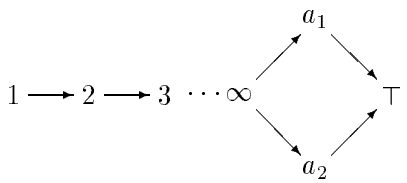
There is another isomorphic description of the canonical basis:

Theorem 8.2 Let X be an algebraic scpo. The canonical basis of $\mathcal{P}_{\mathrm{k}}X$ is isomorphic to the poset of all finite non-empty independent subsets of the canonical basis of X, ordered by Egli-Milner.

Proof: By the uniqueness statement of Prop. 7.7, and by Prop. 7.6 and Lemma 7.9, which yield $\uparrow G \sqsubseteq_{\mathrm{k}} \uparrow H$ iff $G \sqsubseteq_{\mathrm{EM}} H$. □

Surprisingly, our stable power domains turn out to be identical with the so-called *lossless power domains* of [4, 14], which were proposed to model relations in data bases. So far, lossless power domains were never related with stable functions, nor were they shown to be semilattices.

In general, the lossless description of $\mathcal{P}X$ is not very suitable to represent addition. Consider for instance the following algebraic scpo:



The points a_1 and a_2 are isolated, but their compatible meet ∞ is not. Thus, the two sets $\{a_1\}$ and $\{a_2\}$ have no sum in the lossless description directly; instead their sum must be described by the ideal $\{\{1\}, \{2\}, \ldots\}$.

This problem does not occur if we concentrate on *stably algebraic* scpo's.

Definition 8.3 An scpo X is *stably algebraic* iff it is algebraic and its canonical basis is closed w.r.t. compatible meets.

It does not matter whether compatibility of a and b in this definition is understood relative to X or relative to the basis.

Stable algebraicity is preserved by $\mathcal{P}$:

Theorem 8.4 If X is stably algebraic, then so is $\mathcal{P}X$, and its canonical basis is closed w.r.t. addition.

Proof: The second statement holds by Prop. 7.21. It implies stable algebraicity, since compatible meets are instances of sums by Lemma 4.1. □

The proof of Prop. 7.7 indicates how to compute sums in the lossless description if X is stably algebraic: if G and H are non-empty finite independent subsets of the canonical basis of X, then $G + H$ is computed by forming $G \cup H$ and then repeatedly replacing pairs of distinct compatible points of this set by their meet.

9 Domain-theoretic Properties of $\mathcal{P}$

In this section, we consider several classes of scpo's, and investigate whether they are closed under $\mathcal{P}$. If the considered class consists of algebraic scpo's, we can use the lossless description of Theorem 8.2. From Theorems 7.19, 8.1, and 8.4, we already know that $\mathcal{P}$ preserves continuity, algebraicity, and stable algebraicity, and also the ω-versions of these

classes. Here, we study the following properties: property I, finite, discrete, flat, having a least element, property L, bounded complete, distributive, and stably bifinite.

An algebraic dcpo X has *property I* iff the number of points below every isolated point is finite. This property is preserved by $\mathcal{P}$. Note that property I implies stable algebraicity.

Proposition 9.1 If X is an algebraic scpo with property I, then so is $\mathcal{P}X$.

Proof: Let A be a member of the canonical basis of $\mathcal{P}X$, i.e., a non-empty finite set of isolated points of X. The relation $B \sqsubseteq_{\mathrm{EM}} A$ implies $B \subseteq {\downarrow}A$. The set ${\downarrow}A$ is finite as a finite union of finite sets. Thus, the number of subsets of ${\downarrow}A$ is finite. □

A poset is *discrete* iff different points are incomparable: $x \sqsubseteq y$ iff $x = y$. A poset is *flat* iff it consists of a least element $\bot$ and several incomparable points. Flat posets P are denoted by $S_\bot$ where $S = P \setminus \{\bot\}$. All posets that are finite, discrete, or flat are stably algebraic scpo's which coincide with their canonical basis.

Proposition 9.2

(1) If X is finite, then so is $\mathcal{P}X$. It consists of the non-empty independent subsets of X.

(2) If S is a discrete poset, then so is $\mathcal{P}S$. In this case, $\mathcal{P}S$ consists of the finite non-empty subsets of S, and $\mathsf{s}x = \{x\}$ and $A + B = A \cup B$ hold.

(3) If $S_\bot$ is a flat poset, then so is $\mathcal{P}(S_\bot)$, and $\mathcal{P}(S_\bot) \cong (\mathcal{P}S)_\bot$ holds.

Proof: In all three cases, the ground domain coincides with its canonical basis. We apply the lossless description of Theorem 8.2. □

We can show that $\mathcal{P}$ preserves the property to have a least element, because there is a quite obvious 'categorical' description of this property.

Proposition 9.3 For a poset P, the following statements are equivalent:

(1) There is a point $\bot$ in P such that $\bot \sqsubseteq x$ holds for all x in X.

(2) For the one-point poset $\mathbf{1} = \{\diamond\}$, there are monotonic maps $e : \mathbf{1} \to P$ and $r : P \to \mathbf{1}$ such that $e \circ r \sqsubseteq id_P$ holds pointwise.

In case of (2), the least element of P is $e\diamond$. If P is an scpo, then the maps r and e are stable, and $e \circ r \sqsubseteq id_P$ even holds in the stable order.

The criterion above allows proving preservation:

Proposition 9.4 If X is an scpo with least element $\bot$, then $\mathcal{P}X$ has least element $\mathsf{s}\bot$.

Proof: Applying the functor $(\mathcal{P}, \mathsf{M})$ of subsection 3.3, we obtain $\mathsf{M}r : \mathcal{P}X \to \mathcal{P}\mathbf{1}$ and $\mathsf{M}e : \mathcal{P}\mathbf{1} \to \mathcal{P}X$ with $\mathsf{M}e \circ \mathsf{M}r \sqsubseteq id$, where the inequation holds by stability of M. Since $\mathcal{P}\mathbf{1} = \mathcal{P}\{\diamond\} = \{\mathsf{s}\diamond\} \cong \mathbf{1}$ holds by Theorem 9.2, $\mathcal{P}X$ has a least element, namely $\mathsf{M}e(\mathsf{s}\diamond) = \mathsf{s}(e\diamond) = \mathsf{s}\bot_X$. □

Note that we did not assume algebraicity in the proposition above. Thus, it even applies to stable power domains for which no explicit description is known.

A poset P has *property L* iff for every p in P, the set ${\downarrow}\{p\}$ is a complete lattice. For algebraic dcpo's X, property L of X itself is equivalent to property L of the canonical basis of X (see [15]). In the classical case, Plotkin's power construction does not preserve property L (see below). The situation is different in the stable case: In both [4] and [14], it is

indicated that the lossless power domain construction, which is defined for algebraic scpo's only, preserves property L. Thus, we obtain:

Proposition 9.5 If X is an algebraic scpo with property L, then so is $\mathcal{P}X$.

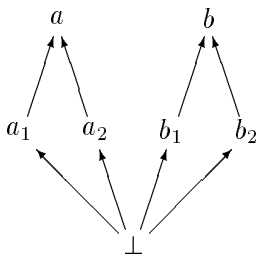
Problem 6 Is property L preserved independently from algebraicity?

In the classical case, the Plotkin power construction does not preserve bounded completeness. The counterexample given in [19] makes also sense in the stable world. It shows that our construction $\mathcal{P}$ neither preserves bounded completeness nor the dI property (algebraic with property I, bounded complete, and distributive).

Let us briefly consider Plotkin's example. Let $X = \mathbf{B} \times \mathbf{B}$, where $\mathbf{B} = \{\bot, \mathsf{T}, \mathsf{F}\}$ is the domain of Booleans. To be concise, we write pairs as xy instead of (x, y), e.g., $\bot\bot$, TF. Let $U = \{\mathsf{T}\bot, \mathsf{F}\bot\}$ and $V = \{\bot\mathsf{T}, \bot\mathsf{F}\}$, and let $Y = \{\mathsf{TT}, \mathsf{FF}\}$ and $Z = \{\mathsf{TF}, \mathsf{FT}\}$. These are non-empty finite independent sets. $U, V \sqsubseteq Y, Z$ holds, but there is nothing in between which could be the join of U and V.

By a slight change, the example also shows that Plotkin's construction does not preserve property L in the classical case. If a greatest element $\top$ is added to X, then $U, V \sqsubseteq Y, Z \sqsubseteq \{\top\}$ holds, but U and V still have no join. This example does not apply in the stable case, because the addition of $\top$ makes the sets U, V, Y, and Z dependent.

Since $\mathcal{P}$ preserves property L connected with algebraicity, it makes sense to ask whether it also preserves distributivity of the complete lattices $\downarrow\{z\}$. This is not the case. Consider the following domain X:



In this domain, every lower cone $\downarrow\{x\}$ is a distributive lattice. Let $C = \{a, b\}$. The sets $\{a_1, b_1\}$, $\{a_2, b_2\}$, and $\{a_1, b_2\}$ are independent and below C. We can compute:

- $\{a_1, b_2\} \sqcap (\{a_1, b_1\} \overset{C}{\sqcup} \{a_2, b_2\}) = \{a_1, b_2\} \sqcap \{a, b\} = \{a_1, b_2\}$;
- $(\{a_1, b_2\} \sqcap \{a_1, b_1\}) \overset{C}{\sqcup} (\{a_1, b_2\} \sqcap \{a_2, b_2\}) = \{\bot\} \overset{C}{\sqcup} \{\bot\} = \{\bot\}$.

This shows the lattice $\downarrow\{C\}$ in $\mathcal{P}X$ is not distributive.

Bifiniteness as it is known from the classical case can be defined in the stable context, too. The definition looks like the classical one [6, 15]:

A function $f : X \to X$ is a *deflation* iff $f \sqsubseteq id$ holds[3] and f has a finite image $f[X]$. A function $f : X \to X$ is *idempotent* iff $f \circ f = f$. An scpo X is *bifinite* iff the identity of X is the join of a directed set of idempotent deflations. X is *ω-bifinite* iff the identity is the join of an ascending sequence of idempotent deflations. The 'bifinites' of [1] correspond to our ω-bifinites with least element.

[3] This refers to the stable order in the stable case.

Although algebraicity is not mentioned in these definitions, every bifinite scpo is stably algebraic with property I [1]. In contrast to the classical case, stable deflations are always idempotent as shown in [1]. Thus, the word 'idempotent' is redundant in the definitions above.

Our goal is to show that the class of bifinite domains is closed under $\mathcal{P}$. Remember that $\mathcal{P}$ forms a functor in the category of scpo's with the map $\mathsf{M} : [X \to Y] \to [\mathcal{P}X \to \mathcal{P}Y]$ of subsection 3.3. The first step is to consider the behavior of a functor operating on functions with finite image.

Proposition 9.6 Let $\mathcal{F}$ be a functor in SCPO that preserves the class of finite scpo's. Then for all scpo's X and Y holds: If $f : X \to Y$ is a morphism with finite image, then $\mathcal{F}f : \mathcal{F}X \to \mathcal{F}Y$ has finite image again.

Proof: Let $Z = \{\sqcap E \mid E \subseteq f[X], \sqcap E \text{ exists}\}$. This is a finite scpo in the order inherited from Y, and the order embedding $e : Z \to Y$ is stable. The original morphism f may be corestricted to $f' : X \to Z$ such that $f = e \circ f'$. Then $\mathcal{F}f = \mathcal{F}(e \circ f') = \mathcal{F}e \circ \mathcal{F}f'$ follows. $\mathcal{F}f'$ maps from $\mathcal{F}X$ to $\mathcal{F}Z$, and the latter is finite because $\mathcal{F}$ preserves finiteness. Thus, the image of $\mathcal{F}f$ is finite. □

The claim of the Proposition is needed to prove the following theorem:

Theorem 9.7 Let $\mathcal{F}$ be a functor in SCPO whose functional part is continuous when considered a family of higher-order functions $\mathcal{F} : [X \to Y] \to [\mathcal{F}X \to \mathcal{F}Y]$. If $\mathcal{F}$ preserves finiteness, then it also preserves bifiniteness and ω-bifiniteness.

Proof: Let X be an (ω-)bifinite domain. Then, there is a (countable) directed set $\mathcal{D}$ of functions from X to X with finite image such that $\sqcup \mathcal{D} = id$. By continuity of $\mathcal{F}$ and Prop. 9.6, $\mathcal{F}[\mathcal{D}]$ is a (countable) directed set of functions from $\mathcal{F}X$ to $\mathcal{F}X$ with finite image, whose join is $\mathcal{F}\,id = id$. Thus, $\mathcal{F}X$ is (ω-)bifinite again. □

Since the functional part of the functor $\mathcal{P}$ is even stable by Prop. 4.6, we can conclude:

Corollary 9.8 If X is (ω-)bifinite, then so is $\mathcal{P}X$.

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