Title: A Cartesian Closed Extension of the Category of Locales
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Summary: We present a Cartesian closed category ELoc of *equilocales*, which contains the category Loc of locales as a reflective full subcategory. The embedding of Loc into ELoc preserves products and all exponentials of exponentiable locales.

More details: So far, no Cartesian-closed extension of the category Loc of locales was known. Here we present one such extension, called the category ELoc of *equilocales*. The new category has some similarity with the category of *equilogical spaces*, which is one of the Cartesian closed extensions of $\mathcal{T}_0$-Top. Recall that there are several equivalent categories of equilogical spaces of different kinds, for instance $\mathcal{T}_0$-topological spaces carrying an equivalence relation, or continuous lattices (= injective spaces) carrying a *partial* equivalence relation (PER). In a similar way, we present two different but equivalent categories of equilocales: the objects of IELoc involve an injective locale and a family of PERs, while the objects of SELoc involve an arbitrary locale and a family of PERs satisfying a joint fullness condition. For matters of economy, we first introduce a larger category ELoc^* whose objects involve an arbitrary locale and a family of PERs.

Note that a PER on a space X in Top, i.e., on the set of points of X, corresponds to a PER on the set $\mathsf{Top}\,(\mathbf{1}, X)$ of continuous functions from the terminal space (one-point space) $\mathbf{1}$ to X. Here, we replace the topological space X by a locale X, but we also need to get away from considering $\mathbf{1}$ since there are non-trivial locales X with no points ($\mathsf{Loc}\,(\mathbf{1}, X) = \emptyset$). The solution is to consider not only a PER on the single set $\mathsf{Loc}\,(\mathbf{1}, X)$, but a family of PERs consisting of one PER on each set $\mathsf{Loc}\,(S, X)$, for any locale S.

Definition: A generalized equilocale (object of ELoc^*) $\mathcal{X}$ is a pair $(X, \sim_{\mathcal{X}})$ consisting of a locale $X = |\mathcal{X}|$ (the *target locale* of $\mathcal{X}$) and a family $\sim_{\mathcal{X}} = (\sim^S_{\mathcal{X}})_{S \in \mathsf{Loc}}$ where $\sim^S_{\mathcal{X}}$ is a PER on the set $\mathsf{Loc}\,(S, X)$ of locale maps from S to X, subject to the following compatibility condition: $\forall\, s : R \to S : \;\; x \sim^S_{\mathcal{X}} x' \;\Rightarrow\; x\,s \sim^R_{\mathcal{X}} x's\,.$

Definition: Given two generalized equilocales $\mathcal{X} = (X, \sim_{\mathcal{X}})$ and $\mathcal{Y} = (Y, \sim_{\mathcal{Y}})$, we define a PER '$\approx$' on the set $\mathsf{Loc}\,(X, Y)$ of locale maps from X to Y by $f \approx f'$ iff for all locales S, $x \sim^S_{\mathcal{X}} x'$ implies $fx \sim^S_{\mathcal{Y}} f'x'$. The set $\mathsf{ELoc}^*\,(\mathcal{X}, \mathcal{Y})$ of ELoc^* maps from $\mathcal{X}$ to $\mathcal{Y}$ is then defined as the set of partial equivalence classes $\mathsf{Loc}\,(X, Y)/{\approx}$.

An *in-equilocale* is a generalized equilocale $(A, \sim_{\mathcal{A}})$ whose target locale A is injective. The full subcategory IELoc of ELoc^* whose objects are in-equilocales is *Cartesian closed.*

A *sur-equilocale* is a generalized equilocale $\mathcal{X} = (X, \sim_{\mathcal{X}})$ such that the class of self-related $x : S \to X$ is jointly epi, i.e., $f\,x = f'\,x$ for all self-related x implies $f = f'$. The full subcategory SELoc of ELoc^* whose objects are sur-equilocales is *equivalent to* IELoc, hence Cartesian closed, too.

The category Loc of locales embeds into SELoc by mapping X to $\widehat{X} = (X, \sim_{\widehat{X}})$ with $x \sim^S_{\widehat{X}} x'$ iff $x = x'$. This embedding preserves products and exponentials Z^Y of exponentiable locales Y. (A locale Y is *exponentiable* if exponentials Z^Y exist for all locales Z.) Finally, we establish a reflection from SELoc to its subcategory Loc.

In showing these results, we never need to delve into the details of the internal structure of locales. We only need some general properties of these objects: products, equalizers, and coequalizers exist, every locale is a sublocale (regular subobject) of an injective locale, and the category of injective locales is Cartesian closed. Thus, our results hold in fact for categories different from Loc if only the required general properties are guaranteed.